Two-dimensional shapes such as quadrilaterals and circles can be used to describe and model the world around us. In this unit, you will learn about the properties of quadrilaterals and circles and how these two-dimensional figures can be transformed.
“Geocaching” Sends Folks on a Scavenger Hunt

“N42 DEGREES 02.054 W88 DEGREES 12.329 – Forget the poison ivy and needle-sharp brambles. Dave April is a man on a mission. Clutching a palm-size Global Positioning System (GPS) receiver in one hand and a computer printout with latitude and longitude coordinates in the other, the 37-year-old software developer trudges doggedly through a suburban Chicago forest preserve, intent on finding a geek’s version of buried treasure.” Geocaching is one of the many new ways that people are spending their leisure time. In this project, you will use quadrilaterals, circles, and geometric transformations to give clues for a treasure hunt.

Log on to www.geometryonline.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 3.
Several different geometric shapes are examples of quadrilaterals. These shapes each have individual characteristics. A rectangle is a type of quadrilateral. Tennis courts are rectangles, and the properties of the rectangular court are used in the game. You will learn more about tennis courts in Lesson 8-4.
Prerequisite Skills  To be successful in this chapter, you’ll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 8.

For Lesson 8-1
Find x for each figure. (For review, see Lesson 4-2.)
1. 
   \[
   \begin{array}{c}
   x' \quad 50^\circ \\
   \end{array}
   \]
2. 
   \[
   x' \quad 25^\circ \\
   \end{array}
   \]
3. 
   \[
   x' \quad 20^\circ \\
   \end{array}
   \]

For Lessons 8-4 and 8-5
Perpendicular Lines
Find the slopes of RS and TS for the given points, R, T, and S. Determine whether RS and TS are perpendicular or not perpendicular. (For review, see Lesson 3-6.)
4. R(4, 3), S(-1, 10), T(13, 20)
5. R(-9, 6), S(3, 8), T(1, 20)
6. R(-6, -1), S(5, 3), T(2, 5)
7. R(-6, 4), S(-3, 8), T(5, 2)

For Lesson 8-7
Slope
Write an expression for the slope of a segment given the coordinates of the endpoints. (For review, see Lesson 3-3.)
8. \((\frac{c}{2}, \frac{d}{2}), (\frac{-c}{2}, \frac{d}{2})\)
9. \((0, a), (b, 0)\)
10. \((-a, c), (-c, a)\)

Foldables Study Organizer

Step 1 Fold
Fold lengthwise to the left margin.

Step 2 Cut
Cut 4 tabs.

Step 3 Label
Label the tabs using the lesson concepts.

Quadrilaterals  Make this Foldable to help you organize your notes. Begin with a sheet of notebook paper.

Reading and Writing  As you read and study the chapter, use your Foldable to take notes, define terms, and record concepts about quadrilaterals.
Angles of Polygons

What You’ll Learn

• Find the sum of the measures of the interior angles of a polygon.
• Find the sum of the measures of the exterior angles of a polygon.

How does a scallop shell illustrate the angles of polygons?

This scallop shell resembles a 12-sided polygon with diagonals drawn from one of the vertices. A diagonal of a polygon is a segment that connects any two nonconsecutive vertices. For example, $\overline{AB}$ is one of the diagonals of this polygon.

SUM OF MEASURES OF INTERIOR ANGLES

Polygons with more than three sides have diagonals. The polygons below show all of the possible diagonals drawn from one vertex.

In each case, the polygon is separated into triangles. Each angle of the polygon is made up of one or more angles of triangles. The sum of the measures of the angles of each polygon can be found by adding the measures of the angles of the triangles. Since the sum of the measures of the angles in a triangle is 180, we can easily find this sum. Make a table to find the sum of the angle measures for several convex polygons.

<table>
<thead>
<tr>
<th>Convex Polygon</th>
<th>Number of Sides</th>
<th>Number of Triangle</th>
<th>Sum of Angle Measures</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>3</td>
<td>1</td>
<td>$(1 \cdot 180)$ or 180</td>
</tr>
<tr>
<td>quadrilateral</td>
<td>4</td>
<td>2</td>
<td>$(2 \cdot 180)$ or 360</td>
</tr>
<tr>
<td>pentagon</td>
<td>5</td>
<td>3</td>
<td>$(3 \cdot 180)$ or 540</td>
</tr>
<tr>
<td>hexagon</td>
<td>6</td>
<td>4</td>
<td>$(4 \cdot 180)$ or 720</td>
</tr>
<tr>
<td>heptagon</td>
<td>7</td>
<td>5</td>
<td>$(5 \cdot 180)$ or 900</td>
</tr>
<tr>
<td>octagon</td>
<td>8</td>
<td>6</td>
<td>$(6 \cdot 180)$ or 1080</td>
</tr>
</tbody>
</table>

Look for a pattern in the sum of the angle measures. In each case, the sum of the angle measures is 2 less than the number of sides in the polygon times 180. So in an $n$-gon, the sum of the angle measures will be $(n - 2)180$ or $180(n - 2)$.

Theorem 8.1

Interior Angle Sum Theorem If a convex polygon has $n$ sides and $S$ is the sum of the measures of its interior angles, then $S = 180(n - 2)$. 

Example:

If $n = 5$ then $S = 180(5 - 2) = 180(3) = 540$. 

Virginia SOL

Standard G.3 The student will solve practical problems involving … angles that include … angles in polygons. Standard G.9 The student will use measures of interior and exterior angles of polygons to solve problems…

Look Back

To review the sum of the measures of the angles of a triangle, see Lesson 4-2.
Lesson 8-1  Angles of Polygons

CHEMISTRY  The benzene molecule, C₆H₆, consists of six carbon atoms in a regular hexagonal pattern with a hydrogen atom attached to each carbon atom. Find the sum of the measures of the interior angles of the hexagon.

Since the molecule is a convex polygon, we can use the Interior Angle Sum Theorem.

\[ S = 180(n - 2) \quad \text{Interior Angle Sum Theorem} \]

\[ = 180(6 - 2) \quad n = 6 \]

\[ = 180(4) \text{ or } 720 \quad \text{Simplify.} \]

The sum of the measures of the interior angles is 720.

The Interior Angle Sum Theorem can also be used to find the number of sides in a regular polygon if you are given the measure of one interior angle.

Example 1  Interior Angles of Regular Polygons

Example 2  Sides of a Polygon

The measure of an interior angle of a regular polygon is 108. Find the number of sides in the polygon.

Use the Interior Angle Sum Theorem to write an equation to solve for \( n \), the number of sides.

\[ S = 180(n - 2) \quad \text{Interior Angle Sum Theorem} \]

\[ (108)n = 180(n - 2) \quad S = 108n \]

\[ 108n = 180n - 360 \quad \text{Distributive Property} \]

\[ 0 = 72n - 360 \quad \text{Subtract } 108n \text{ from each side.} \]

\[ 360 = 72n \quad \text{Add } 360 \text{ to each side.} \]

\[ 5 = n \quad \text{Divide each side by } 72. \]

The polygon has 5 sides.

In Example 2, the Interior Angle Sum Theorem was applied to a regular polygon. In Example 3, we will apply this theorem to a quadrilateral that is not a regular polygon.

Example 3  Interior Angles

ALGEBRA  Find the measure of each interior angle.

Since \( n = 4 \), the sum of the measures of the interior angles is 180(4 - 2) or 360. Write an equation to express the sum of the measures of the interior angles of the polygon.

\[ 360 = m\angle A + m\angle B + m\angle C + m\angle D \quad \text{Sum of measures of angles} \]

\[ 360 = x + 2x + 2x + x \quad \text{Substitution} \]

\[ 360 = 6x \quad \text{Combine like terms.} \]

\[ 60 = x \quad \text{Divide each side by } 6. \]

Use the value of \( x \) to find the measure of each angle.

\[ m\angle A = 60, \ m\angle B = 2 \cdot 60 \text{ or } 120, \ m\angle C = 2 \cdot 60 \text{ or } 120, \text{ and } \ m\angle D = 60. \]
**SUM OF MEASURES OF EXTERIOR ANGLES**
The Interior Angle Sum Theorem relates the interior angles of a convex polygon to the number of sides. Is there a relationship among the exterior angles of a convex polygon?

**Geometry Activity**

**Sum of the Exterior Angles of a Polygon**

**Collect Data**
- Draw a triangle, a convex quadrilateral, a convex pentagon, a convex hexagon, and a convex heptagon.
- Extend the sides of each polygon to form exactly one exterior angle at each vertex.
- Use a protractor to measure each exterior angle of each polygon and record it on your drawing.

**Analyze the Data**
1. Copy and complete the table.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>triangle</th>
<th>quadrilateral</th>
<th>pentagon</th>
<th>hexagon</th>
<th>heptagon</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of exterior angles</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sum of measure of exterior angles</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. What conjecture can you make?

The Geometry Activity suggests Theorem 8.2.

**Theorem 8.2**

**Exterior Angle Sum Theorem**
If a polygon is convex, then the sum of the measures of the exterior angles, one at each vertex, is 360.

**Example:**

\[ m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 = 360 \]

You will prove Theorem 8.2 in Exercise 42.

**Example 4** **Exterior Angles**

Find the measures of an exterior angle and an interior angle of convex regular octagon \( ABCDEFGH \).

At each vertex, extend a side to form one exterior angle. The sum of the measures of the exterior angles is 360. A convex regular octagon has 8 congruent exterior angles.

\[ 8n = 360 \quad n = \text{measure of each exterior angle} \]

\[ n = 45 \quad \text{Divide each side by 8.} \]

The measure of each exterior angle is 45. Since each exterior angle and its corresponding interior angle form a linear pair, the measure of the interior angle is \( 180 - 45 \) or 135.
Practice and Apply

**Concept Check**

1. Explain why the Interior Angle Sum Theorem and the Exterior Angle Sum Theorem only apply to convex polygons.
2. Determine whether the Interior Angle Sum Theorem and the Exterior Angle Sum Theorem apply to polygons that are not regular. Explain.
3. OPEN ENDED Draw a regular convex polygon and a convex polygon that is not regular with the same number of sides. Find the sum of the interior angles for each.

**Guided Practice**

Find the sum of the measures of the interior angles of each convex polygon.

4. pentagon
5. dodecagon

The measure of an interior angle of a regular polygon is given. Find the number of sides in each polygon.

6. 60
7. 90

ALGEBRA Find the measure of each interior angle.

8. \[(3x - 4)\]°
9. \[(9x + 30)\]°

Find the measures of an exterior angle and an interior angle given the number of sides of each regular polygon.

10. 6
11. 18

**Application**

12. AQUARIUMS The regular polygon at the right is the base of a fish tank. Find the sum of the measures of the interior angles of the pentagon.

13. 32-gon
14. 18-gon
15. 19-gon
16. 27-gon
17. 4y-gon
18. 2x-gon

19. GARDENING Carlotta is designing a garden for her backyard. She wants a flower bed shaped like a regular octagon. Find the sum of the measures of the interior angles of the octagon.

20. GAZEBOS A company is building regular hexagonal gazebos. Find the sum of the measures of the interior angles of the hexagon.

The measure of an interior angle of a regular polygon is given. Find the number of sides in each polygon.

21. 140
22. 170
23. 160
24. 165
25. \[157 \frac{1}{2}\]
26. \[176 \frac{2}{5}\]
ALGEBRA  Find the measure of each interior angle using the given information.

27. \[ M \] \[ x \] \[ P \] \[ 5x \] \[ Q \] \[ 2x \]

29. parallelogram \( MNPQ \) with \( m \angle M = 10x \) and \( m \angle N = 20x \)

30. isosceles trapezoid \( TWYZ \) with \( \angle Z \equiv \angle Y \), \( m \angle Z = 30x \), \( \angle T \equiv \angle W \), and \( m \angle T = 20x \)

31. decagon in which the measures of the interior angles are \( x + 5, x + 10, x + 20, x + 30, x + 35, x + 40, x + 60, x + 70, x + 80, \) and \( x + 90 \)

32. polygon \( ABCDE \) with \( m \angle A = 6x \), \( m \angle B = 4x + 13 \), \( m \angle C = x + 9 \), \( m \angle D = 2x - 8 \), and \( m \angle E = 4x - 1 \)

33. quadrilateral in which the measures of the angles are consecutive multiples of \( x \)

34. quadrilateral in which the measure of each consecutive angle increases by 10

Find the measures of each exterior angle and each interior angle for each regular polygon.

35. decagon

36. hexagon

37. nonagon

38. octagon

Find the measures of an interior angle and an exterior angle given the number of sides of each regular polygon. Round to the nearest tenth if necessary.

39. 11  
40. 7  
41. 12

42. PROOF  Use algebra to prove the Exterior Angle Sum Theorem.

43. ARCHITECTURE  The Pentagon building in Washington, D.C., was designed to resemble a regular pentagon. Find the measure of an interior angle and an exterior angle of the courtyard.

44. ARCHITECTURE  Compare the dome to the architectural elements on each side of the dome. Are the interior and exterior angles the same? Find the measures of the interior and exterior angles.

45. CRITICAL THINKING  Two formulas can be used to find the measure of an interior angle of a regular polygon: \( s = \frac{180(n - 2)}{n} \) and \( s = 180 - \frac{360}{n} \). Show that these are equivalent.
46. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

**How does a scallop shell illustrate the angles of polygons?**

Include the following in your answer:
• explain how triangles are related to the Interior Angle Sum Theorem, and
• describe how to find the measure of an exterior angle of a polygon.

47. A regular pentagon and a square share a mutual vertex X. The sides XY and XZ are sides of a third regular polygon with a vertex at X. How many sides does this polygon have?

![Diagram of a regular pentagon and a square sharing a vertex X, with sides XY and XZ as sides of a third regular polygon.]

- A 19  
- B 20  
- C 28  
- D 32

48. **GRID IN** If $6x + 3y = 48$ and $\frac{9y}{2x} = 9$, then $x = ?$

49. In $\triangle ABC$, given the lengths of the sides, find the measure of the given angle to the nearest tenth. **(Lesson 7-7)**

<table>
<thead>
<tr>
<th>Angle</th>
<th>Given Sides</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\angle C$</td>
<td>$a = 6$, $b = 9$, $c = 11$</td>
<td>$11$</td>
</tr>
<tr>
<td>$\angle A$</td>
<td>$a = 47$, $b = 53$, $c = 56$</td>
<td>$72.7$</td>
</tr>
<tr>
<td>$\angle B$</td>
<td>$a = 15.5$, $b = 23.6$, $c = 25.1$</td>
<td>$106.2$</td>
</tr>
<tr>
<td>$\angle C$</td>
<td>$a = 12$, $b = 14$, $c = 16$</td>
<td>$77.1$</td>
</tr>
</tbody>
</table>

50. Solve each $\triangle FGH$ described below. Round angle measures to the nearest degree and side measures to the nearest tenth. **(Lesson 7-6)**

<table>
<thead>
<tr>
<th>Triangle</th>
<th>Side Measures</th>
<th>Angle Measures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\triangle FGH$</td>
<td>$f = 15$, $g = 17$, $m\angle F = 54$</td>
<td>$m\angle G = 65$, $m\angle H = 67$, $g = 63$</td>
</tr>
<tr>
<td>$\triangle FGH$</td>
<td>$m\angle F = 47$, $m\angle H = 78$, $g = 31$</td>
<td>$m\angle G = 65$, $m\angle H = 67$, $g = 63$</td>
</tr>
<tr>
<td>$\triangle FGH$</td>
<td>$g = 30.7$, $h = 32.4$, $m\angle G = 65$</td>
<td>$m\angle F = 47$, $m\angle H = 78$, $g = 31$</td>
</tr>
</tbody>
</table>

51. **PROOF** Write a two-column proof. **(Lesson 4-5)**

**Given:** $\overline{JL} \parallel \overline{KM}$  
$\overline{JK} \parallel \overline{LM}$

**Prove:** $\triangle JKL \cong \triangle MLK$

State the transversal that forms each pair of angles. Then identify the special name for the angle pair. **(Lesson 3-1)**

<table>
<thead>
<tr>
<th>Angle Pair</th>
<th>Transversal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\angle 3$ and $\angle 11$</td>
<td>$\overline{KL}$</td>
</tr>
<tr>
<td>$\angle 6$ and $\angle 7$</td>
<td>$\overline{LM}$</td>
</tr>
<tr>
<td>$\angle 8$ and $\angle 10$</td>
<td>$\overline{JK}$</td>
</tr>
<tr>
<td>$\angle 12$ and $\angle 16$</td>
<td>$\overline{LJ}$</td>
</tr>
</tbody>
</table>

52. **PREREQUISITE SKILL** In the figure, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$. Name all pairs of angles for each type indicated. **(To review angles formed by parallel lines and a transversal, see Lesson 3-1.)**

<table>
<thead>
<tr>
<th>Angle Type</th>
<th>Pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consecutive Interior Angles</td>
<td>$\angle 1, \angle 2$</td>
</tr>
<tr>
<td>Alternate Interior Angles</td>
<td>$\angle 3, \angle 6$</td>
</tr>
<tr>
<td>Corresponding Angles</td>
<td>$\angle 4, \angle 5$</td>
</tr>
<tr>
<td>Alternate Exterior Angles</td>
<td>$\angle 12, \angle 16$</td>
</tr>
</tbody>
</table>
Angles of Polygons

It is possible to find the interior and exterior measurements along with the sum of the interior angles of any regular polygon with \( n \) number of sides using a spreadsheet.

Example

Design a spreadsheet using the following steps.

- Label the columns as shown in the spreadsheet below.
- Enter the digits 3–10 in the first column.
- The number of triangles formed by diagonals from the same vertex in a polygon is 2 less than the number of sides. Write a formula for Cell B2 to subtract 2 from each number in Cell A2.
- Enter a formula for Cell C2 so the spreadsheet will find the sum of the measures of the interior angles. Remember that the formula is \( S = \frac{(n - 2)180}{n} \).
- Continue to enter formulas so that the indicated computation is performed. Then, copy each formula through Row 9. The final spreadsheet will appear as below.

![Spreadsheet Image]

Exercises

1. Write the formula to find the measure of each interior angle in the polygon.
2. Write the formula to find the sum of the measures of the exterior angles.
3. What is the measure of each interior angle if the number of sides is 1? 2?
4. Is it possible to have values of 1 and 2 for the number of sides? Explain.

For Exercises 5–8, use the spreadsheet.

5. How many triangles are in a polygon with 15 sides?
6. Find the measure of the exterior angle of a polygon with 15 sides.
7. Find the measure of the interior angle of a polygon with 110 sides.
8. If the measure of the exterior angles is 0, find the measure of the interior angles. Is this possible? Explain.
8-2 Parallelograms

What You’ll Learn

- Recognize and apply properties of the sides and angles of parallelograms.
- Recognize and apply properties of the diagonals of parallelograms.

Vocabulary

- parallelogram

USA TODAY Snapshots®

Large companies have increased using the Internet to attract and hire employees.

More than three-quarters of Global 500 companies use their Web sites to recruit potential employees:

<table>
<thead>
<tr>
<th>Year</th>
<th>1998</th>
<th>1999</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>79%</td>
<td>60%</td>
<td>29%</td>
<td></td>
</tr>
</tbody>
</table>

SIDES AND ANGLES OF PARALLELOGRAMS

A quadrilateral with parallel opposite sides is called a parallelogram.

Key Concept

**Parallelogram**

- **Symbols** □ABCD
  - There are two pairs of parallel sides. AB and DC, AD and BC

**Example**

- **Example**
  - A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

This activity will help you make conjectures about the sides and angles of a parallelogram.

Geometry Activity

Properties of Parallelograms

Make a model

Step 1 Draw two sets of intersecting parallel lines on patty paper. Label the vertices FGHJ.
**Proof of Theorem 8.4**

Write a two-column proof of Theorem 8.4.

**Given:** \(\square ABCD\)

**Prove:** \(\angle A \equiv \angle C\), \(\angle D \equiv \angle B\)

**Proof:**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (\square ABCD)</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. (AB \parallel DC), (AD \parallel BC)</td>
<td>2. Definition of parallelogram</td>
</tr>
<tr>
<td>3. (\angle A) and (\angle D) are supplementary. (\angle D) and (\angle C) are supplementary. (\angle C) and (\angle B) are supplementary.</td>
<td>3. If parallel lines are cut by a transversal, consecutive interior angles are supplementary.</td>
</tr>
<tr>
<td>4. (\angle A \equiv \angle C), (\angle D \equiv \angle B)</td>
<td>4. Supplements of the same angles are congruent.</td>
</tr>
</tbody>
</table>
**Example 2** Properties of Parallelograms

**ALGEBRA** Quadrilateral $LMNP$ is a parallelogram. Find $m\angle PLM$, $m\angle LMN$, and $d$.

Find $m\angle PLM$, $m\angle LMN$, and $d$.

\[ m\angle PLM = 66 + 42 \text{ or } 108 \]

*Angle Addition Theorem*

\[ \angle PLM = \angle MNP \quad \text{Opp. } \angle \text{ of } \square \text{ are } \equiv. \]

\[ m\angle PLM = 108 \quad \text{Definition of congruent angles} \]

\[ m\angle PLM + m\angle LMN = 180 \quad \text{Cons. } \angle \text{ of } \square \text{ are suppl.} \]

\[ 108 + m\angle LMN = 180 \quad \text{Substitution} \]

\[ m\angle LMN = 72 \quad \text{Subtract 108 from each side.} \]

$LM \equiv PN$ Opp. sides of $\square$ are $\equiv$.

$LM = PN$ Definition of congruent segments

$2d = 22$ Substitution

$d = 11$ Substitution

**DIAGONALS OF PARALLELOGRAMS**

In parallelogram $JKLM$, $JL$ and $KM$ are diagonals. Theorem 8.7 states the relationship between diagonals of a parallelogram.

**Theorem 8.7**

The diagonals of a parallelogram bisect each other.

**Abbreviation:** Diag. of $\square$ bisect each other.

**Example:** $RQ \equiv QT$ and $SQ \equiv QU$

You will prove Theorem 8.7 in Exercise 44.

**Example 3** Diagonals of a Parallelogram

**Multiple-Choice Test Item**

What are the coordinates of the intersection of the diagonals of parallelogram $ABCD$ with vertices $A(2, 5)$, $B(6, 6)$, $C(4, 0)$, and $D(0, -1)$?

- **A** $(4, 2)$
- **B** $(4.5, 2)$
- **C** $(7, -5)$
- **D** $(3, 2.5)$

**Read the Test Item**

Since the diagonals of a parallelogram bisect each other, the intersection point is the midpoint of $AC$ and $BD$.

**Solve the Test Item**

Find the midpoint of $AC$.

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{2 + 4}{2}, \frac{5 + 0}{2} \right) = (3, 2.5)
\]

The coordinates of the intersection of the diagonals of parallelogram $ABCD$ are $(3, 2.5)$. The answer is D.
Theorem 8.8 describes another characteristic of the diagonals of a parallelogram.

Theorem 8.8
Each diagonal of a parallelogram separates the parallelogram into two congruent triangles.

Abbreviation: Diag. separates □ into 2 ≡ Δs.
Example: ΔACD ≡ ΔCAB

You will prove Theorem 8.8 in Exercise 45.

Check for Understanding

Concept Check
1. Describe the characteristics of the sides and angles of a parallelogram.
2. Describe the properties of the diagonals of a parallelogram.
3. OPEN ENDED Draw a parallelogram with one side twice as long as another side.

Guided Practice
Complete each statement about □QRST.
Justify your answer.
4. SV ≡ ?
5. ΔVRS ≡ ?
6. ∠TSR is supplementary to ?.

Use □JKLM to find each measure or value if JK = 2b + 3 and JM = 3a.
7. m∠MJK
8. m∠JML
9. m∠JKL
10. m∠KJL
11. a
12. b

PROOF Write the indicated type of proof.
13. two-column
   Given: □VZRQ and □WQST
   Prove: ∠Z ≡ ∠T
14. paragraph
   Given: □XYZR, WZ ≡ WS
   Prove: ∠XYR ≡ ∠S

Standardized Test Practice
15. MULTIPLE CHOICE Find the coordinates of the intersection of the diagonals of parallelogram GHJK with vertices G(−3, 4), H(1, 1), J(3, −5), and K(−1, −2).
   A. (0, 0.5)  B. (6, −1)  C. (0, −0.5)  D. (5, 0)
Complete each statement about \( \square ABCD \). Justify your answer.

16. \( \angle DAB \cong ? \)
17. \( \angle ABD \cong ? \)
18. \( AB \parallel ? \)
19. \( \overline{BG} \cong ? \)
20. \( \triangle ABD \cong ? \)
21. \( \angle ACD \cong ? \)

**ALGEBRA** Use \( \square MNPR \) to find each measure or value.

22. \( m\angle MNP \)
23. \( m\angle NRP \)
24. \( m\angle RNP \)
25. \( m\angle RMN \)
26. \( m\angle MQN \)
27. \( m\angle MQR \)
28. \( x \)
29. \( y \)
30. \( w \)
31. \( z \)

**DRAWING** For Exercises 32 and 33, use the following information.

The frame of a pantograph is a parallelogram.

32. Find \( x \) and \( EG \) if \( EJ = 2x + 1 \) and \( JG = 3x \).
33. Find \( y \) and \( FH \) if \( HJ = \frac{1}{2}y + 2 \) and \( JF = y - \frac{1}{2} \).

34. **DESIGN** The chest of drawers shown at the right is called Side 2. It was designed by Shiro Kuramata. Describe the properties of parallelograms the artist used to place each drawer pull.

35. **ALGEBRA** Parallelogram \( ABCD \) has diagonals \( \overline{AC} \) and \( \overline{DB} \) that intersect at point \( P \). If \( AP = 3a + 18 \), \( AC = 12a \), \( PB = a + 2b \), and \( PD = 3b + 1 \), find \( a \), \( b \), and \( DB \).

36. **ALGEBRA** In parallelogram \( ABCD \), \( AB = 2x + 5 \), \( m\angle BAC = 2y \), \( m\angle B = 120 \), \( m\angle CAD = 21 \), and \( CD = 21 \). Find \( x \) and \( y \).

**COORDINATE GEOMETRY** For Exercises 37–39, refer to \( \square EFGH \).

37. Use the Distance Formula to verify that the diagonals bisect each other.
38. Determine whether the diagonals of this parallelogram are congruent.
39. Find the slopes of \( \overline{EH} \) and \( \overline{EF} \). Are the consecutive sides perpendicular? Explain.

40. Determine the relationship among \( \overline{ACBX} \), \( \overline{ABYC} \), and \( \overline{ABCZ} \) if \( \triangle XYZ \) is equilateral and \( A \), \( B \), and \( C \) are midpoints of \( \overline{XZ} \), \( \overline{XY} \), and \( \overline{ZY} \), respectively.

**PROOF** Write the indicated type of proof.

41. two-column proof of Theorem 8.3
42. two-column proof of Theorem 8.5
43. paragraph proof of Theorem 8.6
44. paragraph proof of Theorem 8.7
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Maintain Your Skills

Find the sum of the measures of the interior angles of each convex polygon. (Lesson 8-1)

52. 14-gon
53. 22-gon
54. 17-gon
55. 36-gon

Determine whether the Law of Sines or the Law of Cosines should be used to solve each triangle. Then solve each triangle. Round to the nearest tenth. (Lesson 7-7)

56. 57. 58.

59. Find the sum of the first 30 numbers in the outside diagonal of Pascal’s triangle. (Lesson 6-6)

60. Find the sum of the first 70 numbers in the second diagonal.

Getting Ready for the Next Lesson

Critical Thinking Find the ratio of $MS$ to $SP$, given that $MNPQ$ is a parallelogram with $MR = \frac{1}{4} MN$. (Lesson 6-6)

Writing in Math Answer the question that was posed at the beginning of the lesson.

How are parallelograms used to represent data?

Include the following in your answer:

• properties of parallelograms, and

• a display of the data in the graphic with a different parallelogram.

Short Response Two consecutive angles of a parallelogram measure $(3x + 42)\degree$ and $(9x - 18)\degree$. Find the measures of the angles. (Lesson 6-6)

Algebra The perimeter of the rectangle $ABCD$ is equal to $p$ and $x = \frac{y}{5}$. What is the value of $y$ in terms of $p$? (Lesson 6-6)

A $\frac{p}{3}$  B $\frac{5p}{12}$  C $\frac{5p}{8}$  D $\frac{5p}{6}$

Writing in Math S

Mixed Review Find the sum of the measures of the interior angles of each convex polygon. (Lesson 8-1)

52. 14-gon
53. 22-gon
54. 17-gon
55. 36-gon

Determine whether the Law of Sines or the Law of Cosines should be used to solve each triangle. Then solve each triangle. Round to the nearest tenth. (Lesson 7-7)

56. 57. 58.

Use Pascal’s Triangle for Exercises 59 and 60. (Lesson 6-6)

59. Find the sum of the first 30 numbers in the outside diagonal of Pascal’s triangle.

60. Find the sum of the first 70 numbers in the second diagonal.

Getting Ready for the Next Lesson PREREQUISITE SKILL The vertices of a quadrilateral are $A(-5, -2)$, $B(-2, 5)$, $C(2, -2)$, and $D(-1, -9)$. Determine whether each segment is a side or a diagonal of the quadrilateral, and find the slope of each segment. (To review slope, see Lesson 3-3.)

61. $AB$
62. $BD$
63. $CD$
8-3 Tests for Parallelograms

What You’ll Learn

• Recognize the conditions that ensure a quadrilateral is a parallelogram.
• Prove that a set of points forms a parallelogram in the coordinate plane.

How are parallelograms used in architecture?

The roof of the covered bridge appears to be a parallelogram. Each pair of opposite sides looks like they are the same length. How can we know for sure if this shape is really a parallelogram?

CONDITIONS FOR A PARALLELOGRAM By definition, the opposite sides of a parallelogram are parallel. So, if a quadrilateral has each pair of opposite sides parallel it is a parallelogram. Other tests can be used to determine if a quadrilateral is a parallelogram.

Geometry Activity

Testing for a Parallelogram

Model

• Cut two straws to one length and two other straws to a different length.
• Connect the straws by inserting a pipe cleaner in one end of each size of straw to form a quadrilateral like the one shown at the right.
• Shift the sides to form quadrilaterals of different shapes.

Analyze

1. Measure the distance between the opposite sides of the quadrilateral in at least three places. Repeat this process for several figures. What can you conclude about opposite sides?
2. Classify the quadrilaterals that you formed.
3. Compare the measures of pairs of opposite sides.
4. Measure the four angles in several of the quadrilaterals. What relationships do you find?

Make a Conjecture

5. What conditions are necessary to verify that a quadrilateral is a parallelogram?
Write a Proof

**Example 1** Write a Proof

**PROOF** Write a paragraph proof for Theorem 8.10

**Given:** \( \angle A \cong \angle C, \angle B \cong \angle D \)

**Prove:** \(ABCD\) is a parallelogram.

**Paragraph Proof:**

Because two points determine a line, we can draw \( \overline{AC} \). We now have two triangles. We know the sum of the angle measures of a triangle is 180, so the sum of the angle measures of two triangles is 360. Therefore, \( m\angle A + m\angle B + m\angle C + m\angle D = 360 \).

Since \( \angle A \cong \angle C \) and \( \angle B \cong \angle D \), \( m\angle A = m\angle C \) and \( m\angle B = m\angle D \). Substitute to find that \( m\angle A + m\angle A + m\angle B + m\angle D = 360 \), or \( 2(m\angle A) + 2(m\angle B) = 360 \).

Dividing each side of the equation by 2 yields \( m\angle A + m\angle B = 180 \). This means that consecutive angles are supplementary and \( \overline{AD} \parallel \overline{BC} \).

Likewise, \( 2m\angle A + 2m\angle D = 360 \), or \( m\angle A + m\angle D = 180 \). These consecutive supplementary angles verify that \( \overline{AB} \parallel \overline{DC} \). Opposite sides are parallel, so \( ABCD \) is a parallelogram.

**Example 2** Properties of Parallelograms

**ART** Some panels in the sculpture appear to be parallelograms. Describe the information needed to determine whether these panels are parallelograms.

A panel is a parallelogram if both pairs of opposite sides are congruent, or if one pair of opposite sides is congruent and parallel. If the diagonals bisect each other, or if both pairs of opposite angles are congruent, then the panel is a parallelogram.
**Example 3**  
**Properties of Parallelograms**

Determine whether the quadrilateral is a parallelogram. Justify your answer.

Each pair of opposite angles have the same measure. Therefore, they are congruent. If both pairs of opposite angles are congruent, the quadrilateral is a parallelogram.

A quadrilateral is a parallelogram if any one of the following is true.

**Concept Summary**  
**Tests for a Parallelogram**

1. Both pairs of opposite sides are parallel. (Definition)
2. Both pairs of opposite sides are congruent. (Theorem 8.9)
3. Both pairs of opposite angles are congruent. (Theorem 8.10)
4. Diagonals bisect each other. (Theorem 8.11)
5. A pair of opposite sides is both parallel and congruent. (Theorem 8.12)

**Example 4**  
**Find Measures**

**ALGEBRA** Find x and y so that each quadrilateral is a parallelogram.

- **a.**

  Opposite sides of a parallelogram are congruent.

  \[ EF \equiv DG \quad \text{Opp. sides of } \square \text{ are } \equiv. \]

  \[ EF = DG \quad \text{Def. of } \equiv \text{ segments} \]

  \[ 4y = 6y - 42 \quad \text{Substitution} \]

  \[ -2y = -42 \quad \text{Subtract } 6y. \]

  \[ y = 21 \quad \text{Divide by } -2. \]

  So, when \( x \) is 12 and \( y \) is 21, \( DEFG \) is a parallelogram.

- **b.**

  Diagonals in a parallelogram bisect each other.

  \[ QT \equiv TS \quad \text{Opp. sides of } \square \text{ are } \equiv. \]

  \[ QT = TS \quad \text{Def. of } \equiv \text{ segments} \]

  \[ 5y = 2y + 12 \quad \text{Substitution} \]

  \[ 3y = 12 \quad \text{Subtract } 2y. \]

  \[ y = 4 \quad \text{Divide by } 3. \]

  \[ PQRS \text{ is a parallelogram when } x = 7 \text{ and } y = 4. \]
**PARALLELOGRAMS ON THE COORDINATE PLANE**  We can use the Distance Formula and the Slope Formula to determine if a quadrilateral is a parallelogram in the coordinate plane.

**Example 5  Use Slope and Distance**

**COORDINATE GEOMETRY**  Determine whether the figure with the given vertices is a parallelogram. Use the method indicated.

a.  \(A(3, 3), B(8, 2), C(6, -1), D(1, 0)\); Slope Formula

If the opposite sides of a quadrilateral are parallel, then it is a parallelogram.

- slope of \(AB = \frac{2 - 3}{8 - 3} = -\frac{1}{5}\)  
- slope of \(DC = \frac{-1 - 0}{6 - 1} = -\frac{1}{5}\)

- slope of \(AD = \frac{3 - 0}{3 - 1} = \frac{3}{2}\)  
- slope of \(BC = \frac{-2 - 1}{6 - 8} = \frac{3}{2}\)

Since opposite sides have the same slope, \(AB \parallel DC\) and \(AD \parallel BC\). Therefore, \(ABCD\) is a parallelogram by definition.

b.  \(P(5, 3), Q(1, -5), R(-6, -1), S(-2, 7)\); Distance and Slope Formulas

First use the Distance Formula to determine whether the opposite sides are congruent.

- \(PS = \sqrt{(5 - (-2))^2 + (3 - 7)^2}\)  
  - \(= \sqrt{7^2 + (-4)^2} = \sqrt{65}\)

- \(QR = \sqrt{(1 - (-6))^2 + (-5 - (-1))^2}\)  
  - \(= \sqrt{7^2 + (-4)^2} = \sqrt{65}\)

Since \(PS = QR, PS \equiv QR\).

Next, use the Slope Formula to determine whether \(PS \parallel QR\).

- slope of \(PS = \frac{3 - 7}{5 - (-2)} = \frac{4}{7}\)  
- slope of \(QR = \frac{-5 - (-1)}{1 - (-6)} = \frac{4}{7}\)

\(PS\) and \(QR\) have the same slope, so they are parallel. Since one pair of opposite sides is congruent and parallel, \(PQRS\) is a parallelogram.

**Check for Understanding**

**Concept Check**

1. List and describe four tests for parallelograms.
2. **OPEN ENDED**  Draw a parallelogram. Label the congruent angles.
3. **FIND THE ERROR**  Carter and Shaniqua are describing ways to show that a quadrilateral is a parallelogram.

   **Carter**

   A quadrilateral is a parallelogram if one pair of opposite sides is congruent and one pair of opposite sides is parallel.

   **Shaniqua**

   A quadrilateral is a parallelogram if one pair of opposite sides is congruent and parallel.

Who is correct? Explain your reasoning.
Determine whether each quadrilateral is a parallelogram. Justify your answer.

4. 5.

ALGEBRA Find \(x\) and \(y\) so that each quadrilateral is a parallelogram.

6. 7.

COORDINATE GEOMETRY Determine whether the figure with the given vertices is a parallelogram. Use the method indicated.

8. \((0, 0), (4, 1), (6, 5), (2, 4)\); Slope Formula
9. \((-4, 0), (3, 1), (1, 4), (-6, 3)\); Distance and Slope Formulas
10. \((-4, -3), (4, -1), (2, 3), (-6, 2)\); Midpoint Formula

11. PROOF Write a two-column proof to prove that \(PQRS\) is a parallelogram given that \(\overline{PT} \parallel \overline{TR}\) and \(\angle TSP \equiv \angle TQR\).

12. TANGRAMS A tangram set consists of seven pieces: a small square, two small congruent right triangles, two large congruent right triangles, a medium-sized right triangle, and a quadrilateral. How can you determine the shape of the quadrilateral? Explain.

Practice and Apply

Determine whether each quadrilateral is a parallelogram. Justify your answer.

16. 17. 18.

ALGEBRA Find \(x\) and \(y\) so that each quadrilateral is a parallelogram.

19. 20. 21.

22. 23. 24.
COORDINATE GEOMETRY  Determine whether a figure with the given vertices is a parallelogram. Use the method indicated.

25. \( B(-6, -3), C(2, -3), E(4, 4), G(-4, 4); \) Slope Formula
26. \( Q(-3, -6), R(2, 2), S(-1, 6), T(-5, 2); \) Slope Formula
27. \( A(-5, -4), B(3, -2), C(4, 4), D(-4, 2); \) Distance Formula
28. \( W(-6, -5), X(-1, -4), Y(0, -1), Z(-5, -2); \) Midpoint Formula
29. \( G(-2, 8), H(4, 4), J(6, -3), K(-1, -7); \) Distance and Slope Formulas
30. \( H(5, 6), J(9, 0), K(8, -5), L(3, -2); \) Distance Formula
31. \( S(-1, 9), T(3, 8), V(6, 2), W(2, 3); \) Midpoint Formula
32. \( C(-7, 3), D(-3, 2), F(0, -4), G(-4, -3); \) Distance and Slope Formulas

33. Quadrilateral \( MNPR \) has vertices \( M(-6, 6), N(-1, -1), P(-2, -4), \) and \( R(-5, -2). \) Determine how to move one vertex to make \( MNPR \) a parallelogram.

34. Quadrilateral \( QSTW \) has vertices \( Q(-3, 3), S(4, 1), T(-1, -2), \) and \( W(-5, -1). \) Determine how to move one vertex to make \( QSTW \) a parallelogram.

COORDINATE GEOMETRY  The coordinates of three of the vertices of a parallelogram are given. Find the possible coordinates for the fourth vertex.

35. \( A(1, 4), B(7, 5), \) and \( C(4, -1). \)
36. \( Q(-2, 2), R(1, 1), \) and \( S(-1, -1). \)

37. STORAGE  Songan purchased an expandable hat rack that has 11 pegs. In the figure, \( H \) is the midpoint of \( KM \) and \( JL. \) What type of figure is \( JKLM? \) Explain.

38. METEOROLOGY  To show the center of a storm, television stations superimpose a “watchbox” over the weather map. Describe how you know that the watchbox is a parallelogram.

Online Research  Data Update  Each hurricane is assigned a name as the storm develops. What is the name of the most recent hurricane or tropical storm in the Atlantic or Pacific Oceans? Visit www.geometryonline.com/data_update to learn more.

**PROOF**  Write a two-column proof of each theorem.

39. Theorem 8.9  
40. Theorem 8.11  
41. Theorem 8.12

42. Li-Cheng claims she invented a new geometry theorem. \( A \) diagonal of a parallelogram bisects its angles. Determine whether this theorem is true. Find an example or counterexample.

43. CRITICAL THINKING  Write a proof to prove that \( FDCA \) is a parallelogram if \( ABCDEF \) is a regular hexagon.

44. WRITING IN MATH  Answer the question that was posed at the beginning of the lesson.

How are parallelograms used in architecture?

Include the following in your answer:
- the information needed to prove that the roof of the covered bridge is a parallelogram, and
- another example of parallelograms used in architecture.
Practice Quiz 1

1. The measure of an interior angle of a regular polygon is $147\frac{3}{11}^\circ$.
   Find the number of sides in the polygon.  
   (Lesson 8-1)

Use $\square WXYZ$ to find each measure.  
   (Lesson 8-2)

2. $WZ = \ ?$

3. $m\angle XYZ = \ ?$

ALGEBRA Find $x$ and $y$ so that each quadrilateral is a parallelogram.  
   (Lesson 8-3)

4. $\angle (3y + 9)^\circ$, $\angle (3y + 36)^\circ$

5. $2x - 4$, $4y - 8$

Maintain Your Skills

Mixed Review  
Use $\square NQRM$ to find each measure or value.  
   (Lesson 8-2)

47. $w$

48. $x$

49. $NQ$

50. $QR$

The measure of an interior angle of a regular polygon is given. Find the number of sides in each polygon.  
   (Lesson 8-1)

51. 135

52. 144

53. 168

54. 162

55. 175

56. 175.5

Find $x$ and $y$.  
   (Lesson 7-3)

57.

58.

59.

Getting Ready for the Next Lesson

PREREQUISITE SKILL Use slope to determine whether $\overline{AB}$ and $\overline{BC}$ are perpendicular or not perpendicular.  
   (To review slope and perpendicularity, see Lesson 3-3.)

60. $A(2, 5), B(6, 3), C(8, 7)$

61. $A(-1, 2), B(0, 7), C(4, 1)$

62. $A(0, 4), B(5, 7), C(8, 3)$

63. $A(-2, -5), B(1, -3), C(-1, 0)$

Practice Quiz 1

Lessons 8-1 through 8-3

1. The measure of an interior angle of a regular polygon is $147\frac{3}{11}^\circ$.
   Find the number of sides in the polygon.  
   (Lesson 8-1)

Use $\square WXYZ$ to find each measure.  
   (Lesson 8-2)

2. $WZ = \ ?$

3. $m\angle XYZ = \ ?$

ALGEBRA Find $x$ and $y$ so that each quadrilateral is a parallelogram.  
   (Lesson 8-3)

4. $\angle (3y + 9)^\circ$, $\angle (3y + 36)^\circ$

5. $2x - 4$, $4y - 8$
Rectangles

Vocabulary
- rectangle

How are rectangles used in tennis?
Many sports are played on fields marked by parallel lines. A tennis court has parallel lines at half-court for each player. Parallel lines divide the court for singles and doubles play. The service box is marked by perpendicular lines.

**PROPERTIES OF RECTANGLES**
A rectangle is a quadrilateral with four right angles. Since both pairs of opposite angles are congruent, it follows that it is a special type of parallelogram. Thus, a rectangle has all the properties of a parallelogram. Because the right angles make a rectangle a rigid figure, the diagonals are also congruent.

**Theorem 8.13**
If a parallelogram is a rectangle, then the diagonals are congruent.
Abbreviation: If \( \square \) is rectangle, diag. are \( \cong \).

\[ AB \cong DC \hspace{1cm} AB \parallel DC \]
\[ BC \cong AD \hspace{1cm} BC \parallel AD \]

You will prove Theorem 8.13 in Exercise 40.

If a quadrilateral is a rectangle, then the following properties are true.

**Key Concept**

<table>
<thead>
<tr>
<th>Properties</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Opposite sides are congruent and parallel.</td>
<td>( AB \cong DC ) \hspace{1cm} ( AB \parallel DC ) \hspace{1cm} ( BC \cong AD ) \hspace{1cm} ( BC \parallel AD )</td>
</tr>
<tr>
<td>2. Opposite angles are congruent.</td>
<td>( \angle A \cong \angle C ) \hspace{1cm} ( \angle B \cong \angle D )</td>
</tr>
<tr>
<td>3. Consecutive angles are supplementary.</td>
<td>( m\angle A + m\angle B = 180 ) \hspace{1cm} ( m\angle B + m\angle C = 180 ) \hspace{1cm} ( m\angle C + m\angle D = 180 ) \hspace{1cm} ( m\angle D + m\angle A = 180 )</td>
</tr>
<tr>
<td>4. Diagonals are congruent and bisect each other.</td>
<td>( AC ) and ( BD ) bisect each other. \hspace{1cm} ( AC \cong BD )</td>
</tr>
<tr>
<td>5. All four angles are right angles.</td>
<td>( m\angle DAB = m\angle BCD = 90 ) \hspace{1cm} ( m\angle ABC = m\angle ADC = 90 )</td>
</tr>
</tbody>
</table>

**Virginia SOL**
- Standard G.8.a: The student will investigate and identify properties of quadrilaterals involving opposite sides and angles, consecutive sides and angles, and diagonals.
- Standard G.8.b: The student will prove these properties of quadrilaterals, using algebraic ... methods as well as deductive reasoning; and
- Standard G.8.c: The student will use properties of quadrilaterals to solve practical problems.
**Example 1** Diagonals of a Rectangle

**ALGEBRA** Quadrilateral $M NOP$ is a rectangle. If $MO = 6x + 14$ and $PN = 9x + 5$, find $x$.

The diagonals of a rectangle are congruent, so $MO \equiv PN$.

\[
\begin{align*}
MO & \equiv PN \quad \text{Diagonals of a rectangle are } \equiv. \\
MO &= PN \\
6x + 14 &= 9x + 5 \\
14 &= 3x + 5 \quad \text{Substitution} \\
9 &= 3x \quad \text{Subtract 6 from each side.} \\
3 &= x \quad \text{Subtract 5 from each side.} \\
& \text{Divide each side by 3.}
\end{align*}
\]

Rectangles can be constructed using perpendicular lines.

**Construction**

**Rectangle**

1. Use a straightedge to draw line $\ell$. Label a point $P$ on $\ell$. Place the point at $P$ and locate point $Q$ on $\ell$. Now construct lines perpendicular to $\ell$ through $P$ and through $Q$. Label them $m$ and $n$.

2. Place the compass point at $P$ and mark off a segment on $m$. Using the same compass setting, place the compass at $Q$ and mark a segment on $n$. Label these points $R$ and $S$. Draw $RS$.

3. Locate the compass setting that represents $PR$ and compare to the setting for $QS$. The measures should be the same.

**Example 2** Angles of a Rectangle

**ALGEBRA** Quadrilateral $ABCD$ is a rectangle.

a. Find $x$.

$\angle DAB$ is a right angle, so $m\angle DAB = 90$.

\[
m\angle DAC + m\angle BAC = m\angle DAB \\
4x + 5 + 9x + 20 = 90 \\
13x + 25 = 90 \\
13x = 65 \\
x = 5
\]

www.geometryonline.com/extra_examples/sol

Lesson 8-4 Rectangles 425
Diagonals of a Parallelogram

Trent is building a tree house for his younger brother. He has measured the window opening to be sure that the opposite sides are congruent. He measures the diagonals to make sure that they are congruent. This is called squaring the frame. How does he know that the corners are 90° angles?

First draw a diagram and label the vertices. We know that $\overline{WX} \cong \overline{ZY}$, $\overline{XY} \cong \overline{WZ}$, and $\overline{WY} \cong \overline{XZ}$.

Because $\overline{WX} \cong \overline{ZY}$ and $\overline{XY} \cong \overline{WZ}$, $WXYZ$ is a parallelogram.

$XZ$ and $WY$ are diagonals and they are congruent. A parallelogram with congruent diagonals is a rectangle. So, the corners are 90° angles.

PROVE THAT PARALLELOGRAMS ARE RECTANGLES

The converse of Theorem 8.13 is also true.

**Theorem 8.14**

If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Abbreviation: If diagonals of $\square$ are $\cong$, $\square$ is a rectangle.

You will prove Theorem 8.14 in Exercise 41.

**Example 3 Diagonals of a Parallelogram**

**WINDOWS** Trent is building a tree house for his younger brother. He has measured the window opening to be sure that the opposite sides are congruent. He measures the diagonals to make sure that they are congruent. This is called squaring the frame. How does he know that the corners are 90° angles?

First draw a diagram and label the vertices. We know that $\overline{WX} \cong \overline{ZY}$, $\overline{XY} \cong \overline{WZ}$, and $\overline{WY} \cong \overline{XZ}$.

Because $\overline{WX} \cong \overline{ZY}$ and $\overline{XY} \cong \overline{WZ}$, $WXYZ$ is a parallelogram.

$XZ$ and $WY$ are diagonals and they are congruent. A parallelogram with congruent diagonals is a rectangle. So, the corners are 90° angles.

**Example 4 Rectangle on a Coordinate Plane**

**COORDINATE GEOMETRY** Quadrilateral $FGHJ$ has vertices $F(-4, -1)$, $G(-2, -5)$, $H(4, -2)$, and $J(2, 2)$. Determine whether $FGHJ$ is a rectangle.

**Method 1** Use the Slope Formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, to see if consecutive sides are perpendicular.

slope of $\overline{FJ} = \frac{2 - (-1)}{2 - (-4)}$ or $\frac{1}{2}$. 

b. Find $y$.

Since a rectangle is a parallelogram, opposite sides are parallel. So, alternate interior angles are congruent.

\[
\angle ADB \cong \angle CBD \quad \text{Alternate Interior Angles Theorem}
\]

\[
m\angle ADB = m\angle CBD \quad \text{Definition of } \cong \text{ angles}
\]

\[
y^2 - 1 = 4y + 4 \quad \text{Substitution}
\]

\[
y^2 - 4y - 5 = 0 \quad \text{Subtract } 4y \text{ and } 4 \text{ from each side.}
\]

\[
(y - 5)(y + 1) = 0 \quad \text{Factor.}
\]

\[
y - 5 = 0 \quad y + 1 = 0
\]

\[
y = 5 \quad y = -1
\]

Disregard $y = -1$ because it yields angle measures of 0.

Windows

It is important to square the window frame because over time the opening may have become “out-of-square.” If the window is not properly situated in the framed opening, air and moisture can leak through cracks.

Source: www.supersealwindows.com/guide/measurement
Concept Check

1. **OPEN ENDED** How can you determine whether a parallelogram is a rectangle?

2. **FIND THE ERROR** McKenna and Consuelo are defining a rectangle for an assignment.

   **McKenna**
   A rectangle is a parallelogram with one right angle.

   **Consuelo**
   A rectangle has a pair of parallel opposite sides and a right angle.

Who is correct? Explain.

---

slope of \( \overline{GH} = \frac{-2 - (-5)}{4 - (-2)} \) or \( \frac{1}{2} \)

slope of \( \overline{FG} = \frac{-5 - (-1)}{-2 - (-4)} \) or \(-2\)

slope of \( \overline{JH} = \frac{2 - (-2)}{2 - 4} \) or \(-2\)

Because \( \overline{FJ} \parallel \overline{GH} \) and \( \overline{FG} \parallel \overline{JH} \), quadrilateral \( \overline{FGHJ} \) is a parallelogram.

The product of the slopes of consecutive sides is \(-1\). This means that \( \overline{FJ} \perp \overline{FG} \), \( \overline{FJ} \perp \overline{JH} \), \( \overline{JH} \perp \overline{GH} \), and \( \overline{FG} \perp \overline{GH} \). The perpendicular segments create four right angles. Therefore, by definition \( \overline{FGHJ} \) is a rectangle.

**Method 2:** Use the Distance Formula, \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \), to determine whether opposite sides are congruent.

First, we must show that quadrilateral \( \overline{FGHJ} \) is a parallelogram.

\[
\overline{FJ} = \sqrt{(-4 - 2)^2 + (-1 - 2)^2} = \sqrt{36 + 9} = \sqrt{45}
\]

\[
\overline{GH} = \sqrt{(-2 - 4)^2 + [-5 - (-2)]^2} = \sqrt{36 + 9} = \sqrt{45}
\]

\[
\overline{FG} = \sqrt{[-4 - (-2)]^2 + [-1 - (-5)]^2} = \sqrt{4 + 16} = \sqrt{20}
\]

\[
\overline{JH} = \sqrt{(2 - 4)^2 + [2 - (-2)]^2} = \sqrt{4 + 16} = \sqrt{20}
\]

Since each pair of opposite sides of the quadrilateral have the same measure, they are congruent. Quadrilateral \( \overline{FGHJ} \) is a parallelogram.

\[
\overline{FH} = \sqrt{(-4 - 4)^2 + [-1 - (-2)]^2} = \sqrt{64 + 1} = \sqrt{65}
\]

\[
\overline{GJ} = \sqrt{(-2 - 2)^2 + (-5 - 2)^2} = \sqrt{16 + 49} = \sqrt{65}
\]

The length of each diagonal is \( \sqrt{65} \). Since the diagonals are congruent, \( \overline{FGHJ} \) is a rectangle by Theorem 8.14.

---

**Check for Understanding**

**Concept Check**

1. How can you determine whether a parallelogram is a rectangle?

2. **OPEN ENDED** Draw two congruent right triangles with a common hypotenuse. Do the legs form a rectangle?

3. **FIND THE ERROR** McKenna and Consuelo are defining a rectangle for an assignment.

   **McKenna**
   A rectangle is a parallelogram with one right angle.

   **Consuelo**
   A rectangle has a pair of parallel opposite sides and a right angle.

Who is correct? Explain.
4. **ALGEBRA**  
*ABCD* is a rectangle.  
If $AC = 30 - x$ and $BD = 4x - 60$, find $x$.

5. **ALGEBRA**  
*MNQR* is a rectangle.  
If $NR = 2x + 10$ and $NP = 2x - 30$, find $MP$.

6. $x$  
7. $m\angle RPS$  

8. **COORDINATE GEOMETRY**  
Quadrilateral *EFGH* has vertices $E(-4, -3)$, $F(3, -1)$, $G(2, 3)$, and $H(-5, 1)$. Determine whether *EFGH* is a rectangle.

9. **FRAMING**  
Mrs. Walker has a rectangular picture that is 12 inches by 48 inches. Because this is not a standard size, a special frame must be built. What can the framer do to guarantee that the frame is a rectangle? Justify your reasoning.

10. If $NQ = 5x - 3$ and $QM = 4x + 6$, find $NK$.  
11. If $NQ = 2x + 3$ and $QK = 5x - 9$, find $JQ$.  
12. If $NM = 8x - 14$ and $JK = x^2 + 1$, find $JK$.  
13. If $m\angle NJM = 2x - 3$ and $m\angle KJM = x + 5$, find $x$.  
14. If $m\angle NKM = x^2 + 4$ and $m\angle KNM = x + 30$, find $m\angle JKN$.  
15. If $m\angle JKN = 2x^2 + 2$ and $m\angle NKM = 14x$, find $x$.

16. $m\angle 1$  
17. $m\angle 2$  
18. $m\angle 3$  
19. $m\angle 4$  
20. $m\angle 5$  
21. $m\angle 6$  
22. $m\angle 7$  
23. $m\angle 8$  
24. $m\angle 9$

25. **PATIOS**  
A contractor has been hired to pour a rectangular concrete patio. How can he be sure that the frame in which to pour the concrete is rectangular?

26. **TELEVISION**  
Television screens are measured on the diagonal. What is the measure of the diagonal of this screen?
Determine whether $DFGH$ is a rectangle given each set of vertices. Justify your answer.

27. $D(9, -1), F(9, 5), G(-6, 5), H(-6, 1)$
28. $D(6, 2), F(8, -1), G(10, 6), H(12, 3)$
29. $D(-4, -3), F(-5, 8), G(6, 9), H(7, -2)$

The vertices of $WXYZ$ are $W(2, 4), X(-2, 0), Y(-1, -7), and Z(9, 3)$.

Find $WY$ and $XZ$.

Find the coordinates of the midpoints of $WX$ and $YZ$.

Is $WXYZ$ a rectangle? Explain.

The vertices of parallelogram $ABCD$ are $A(-4, -4), B(2, -1), C(0, 3), and D(-6, 0)$.

Determine whether $ABCD$ is a rectangle.

If $ABCD$ is a rectangle and $E, F, G, and H$ are midpoints of its sides, what can you conclude about $EFGH$?

The windmill section of a miniature golf course will be a rectangle 10 feet long and 6 feet wide. Suppose the contractor placed stakes and strings to mark the boundaries with the corners at $A, B, C, and D$. The contractor measured $BD$ and $AC$ and found that $AC > BD$. Describe where to move the stakes $L$ and $K$ to make $ABCD$ a rectangle. Explain.

Many artists have used golden rectangles in their work. In a golden rectangle, the ratio of the length to the width is about 1.618. This ratio is known as the golden ratio.

A rectangle has dimensions of 19.42 feet and 12.01 feet. Determine if the rectangle is a golden rectangle. Then find the length of the diagonal.

Use the Internet or other sources to find examples of golden rectangles.

What are the minimal requirements to justify that a parallelogram is a rectangle?

Draw a counterexample to the statement $If$ the diagonals are congruent, the quadrilateral is a rectangle.

Write a two-column proof.

$PQST$ is a rectangle.

$QR = VT$

$PR = VS$

$DEAC$ and $FEAB$ are rectangles.

$\angle GKH = \angle JHK$

$GJ$ and $HK$ intersect at $L$.

$GHJK$ is a parallelogram.

Using four of the twelve points as corners, how many rectangles can be drawn?
SPHERICAL GEOMETRY  The figure shows a Saccheri quadrilateral on a sphere. Note that it has four sides with $CT \parallel TR$, $AR \perp TR$, and $CT \equiv AR$.

45. Is $CT$ parallel to $AR$? Explain.

46. How does $AC$ compare to $TR$?

47. Can a rectangle exist in spherical geometry? Explain.

48. **WRITING IN MATH**  Answer the question that was posed at the beginning of the lesson.

**How are rectangles used in tennis?**

Include the following in your answer:
- the number of rectangles on one side of a tennis court, and
- a method to ensure the lines on the court are parallel

49. In the figure, $AB \parallel CE$. If $DA = 6$, what is $DB$?

   - A 6
   - B 7
   - C 8
   - D 9

50. **ALGEBRA**  A rectangular playground is surrounded by an 80-foot long fence. One side of the playground is 10 feet longer than the other. Which of the following equations could be used to find $s$, the shorter side of the playground?

   - A $10s + s = 80$
   - B $4s + 10 = 80$
   - C $s(s + 10) = 80$
   - D $2(s + 10) + 2s = 80$

**Maintain Your Skills**

51. **TEXTILE ARTS**  The Navajo people are well known for their skill in weaving. The design at the right, known as the Eye-Dazzler, became popular with Navajo weavers in the 1880s. How many parallelograms, not including rectangles, are in the pattern?  **(Lesson 8-3)**

For Exercises 52–57, use $\square ABCD$. Find each measure or value.  **(Lesson 8-2)**

52. $m\angle AFD$
53. $m\angle CDF$
54. $m\angle FBC$
55. $m\angle BCF$
56. $y$
57. $x$

Find the measure of the altitude drawn to the hypotenuse.  **(Lesson 7-1)**

58.

59.

60.

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL**  Find the distance between each pair of points.  *(To review the Distance Formula, see Lesson 1-4.)*

61. $(1, −2), (−3, 1)$
62. $(−5, 9), (5, 12)$
63. $(1, 4), (22, 24)$
Rhombi and Squares

What You’ll Learn

• Recognize and apply the properties of rhombi.
• Recognize and apply the properties of squares.

How can you ride a bicycle with square wheels?

Professor Stan Wagon at Macalester College in St. Paul, Minnesota, developed a bicycle with square wheels. There are two front wheels so the rider can balance without turning the handlebars. Riding over a specially curved road ensures a smooth ride.

PROPERTIES OF RHOMBI A square is a special type of parallelogram called a rhombus. A rhombus is a quadrilateral with all four sides congruent. All of the properties of parallelograms can be applied to rhombi. There are three other characteristics of rhombi described in the following theorems.

Key Concept

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.15</td>
<td>The diagonals of a rhombus are perpendicular. $\overline{AC} \perp \overline{BD}$</td>
</tr>
<tr>
<td>8.16</td>
<td>If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus. (Converse of Theorem 8.15) If $\overline{BD} \perp \overline{AC}$, then $\Box ABCD$ is a rhombus.</td>
</tr>
<tr>
<td>8.17</td>
<td>Each diagonal of a rhombus bisects a pair of opposite angles. $\angle DAC \equiv \angle BAC \equiv \angle DCA \equiv \angle BCA \equiv \angle ABD \equiv \angle CBD \equiv \angle ADB \equiv \angle CDB$</td>
</tr>
</tbody>
</table>

You will prove Theorems 8.16 and 8.17 in Exercises 35 and 36, respectively.

Example 1 Proof of Theorem 8.15

Given: $PQRS$ is a rhombus. $P \overline{R} \perp \overline{SQ}$

Proof: $\overline{PQ} \equiv \overline{QR} \equiv \overline{RS} \equiv \overline{PS}$. A rhombus is a parallelogram and the diagonals of a parallelogram bisect each other, so $\overline{QS}$ bisects $\overline{PR}$ at $T$. Thus, $\overline{PT} \equiv \overline{RT}$. $\overline{QT} \equiv \overline{QT}$ because congruence of segments is reflexive. Thus, $\triangle PQT \equiv \triangle RQT$ by SSS. $\angle QTP \equiv \angle QTR$ by CPCTC. $\angle QTP$ and $\angle QTR$ also form a linear pair. Two congruent angles that form a linear pair are right angles. $\angle QTP$ is a right angle, so $\overline{PR} \perp \overline{SQ}$ by the definition of perpendicular lines.

Study Tip

Virginia SOL Standard G.8.a The student will investigate and identify properties of quadrilaterals involving opposite sides and angles, consecutive sides and angles, and diagonals. Standard G.8.b The student will prove these properties of quadrilaterals, using algebraic … methods as well as deductive reasoning; and Standard G.8.c The student will use properties of quadrilaterals to solve practical problems.

Professors Stan Wagon at Macalester College showed how to design a bicycle with square wheels.
**Example 2** Measures of a Rhombus

**ALGEBRA** Use rhombus $QRST$ and the given information to find the value of each variable.

a. Find $y$ if $m\angle 3 = y^2 - 31$.

$m\angle 3 = 90$  \hspace{1em}  The diagonals of a rhombus are perpendicular.

$y^2 - 31 = 90$  \hspace{1em}  Substitution

$y^2 = 121$  \hspace{1em}  Add 31 to each side.

$y = \pm 11$  \hspace{1em}  Take the square root of each side.

The value of $y$ can be 11 or $-11$.

b. Find $m\angle TQS$ if $m\angle RST = 56$.

$m\angle TQR = m\angle RST$  \hspace{1em}  Opposite angles are congruent.

$m\angle TQR = 56$  \hspace{1em}  Substitution

The diagonals of a rhombus bisect the angles. So, $m\angle TQS = \frac{1}{2}(56)$ or 28.

**Properties of Squares** If a quadrilateral is both a rhombus and a rectangle, then it is a square. All of the properties of parallelograms and rectangles can be applied to squares.

**Example 3** Squares

**Coordinate Geometry** Determine whether parallelogram $ABCD$ is a rhombus, a rectangle, or a square. List all that apply. Explain.

**Explore** Plot the vertices on a coordinate plane.

**Plan** If the diagonals are perpendicular, then $ABCD$ is either a rhombus or a square. The diagonals of a rectangle are congruent. If the diagonals are congruent and perpendicular, then $ABCD$ is a square.

**Solve** Use the Distance Formula to compare the lengths of the diagonals.

$DB = \sqrt{(3 - (-3))^2 + (-1 - 1)^2}$

$AC = \sqrt{(1 + 1)^2 + (3 + 3)^2}$

$= \sqrt{36 + 4}$

$= \sqrt{40}$

$= \sqrt{4 + 36}$

$= \sqrt{40}$

Use slope to determine whether the diagonals are perpendicular.

$slope \ of \ DB = \frac{1 - (-1)}{-3 - 3} \ or \ -\frac{1}{3}$  \hspace{1em}  slope \ of \ $AC = \frac{-3 - 3}{-1 - 1} \ or \ 3$

Since the slope of $AC$ is the negative reciprocal of the slope of $DB$, the diagonals are perpendicular. The lengths of $DB$ and $AC$ are the same so the diagonals are congruent. $ABCD$ is a rhombus, a rectangle, and a square.

**Examine** You can verify that $ABCD$ is a square by finding the measure and slope of each side. All four sides are congruent and consecutive sides are perpendicular.
If a quadrilateral is a rhombus or a square, then the following properties are true.

### Diagonals of a Square

**BASEBALL** The infield of a baseball diamond is a square, as shown at the right. Is the pitcher’s mound located in the center of the infield? Explain.

Since a square is a parallelogram, the diagonals bisect each other. Since a square is a rhombus, the diagonals are congruent. Therefore, the distance from first base to third base is equal to the distance between home plate and second base.

Thus, the distance from home plate to the center of the infield is 127 feet $3\frac{3}{8}$ inches divided by 2 or 63 feet $7\frac{11}{16}$ inches. This distance is longer than the distance from home plate to the pitcher’s mound so the pitcher’s mound is not located in the center of the field. It is about 3 feet closer to home.

If a quadrilateral is a rhombus or a square, then the following properties are true.

### Concept Summary

<table>
<thead>
<tr>
<th><strong>Rhombi</strong></th>
<th><strong>Squares</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. A rhombus has all the properties of a parallelogram.</td>
<td>1. A square has all the properties of a parallelogram.</td>
</tr>
<tr>
<td>2. All sides are congruent.</td>
<td>2. A square has all the properties of a rectangle.</td>
</tr>
<tr>
<td>3. Diagonals are perpendicular.</td>
<td>3. A square has all the properties of a rhombus.</td>
</tr>
<tr>
<td>4. Diagonals bisect the angles of the rhombus.</td>
<td></td>
</tr>
</tbody>
</table>
1. Draw a diagram to demonstrate the relationship among parallelograms, rectangles, rhombi, and squares.

2. OPEN ENDED Draw a quadrilateral that has the characteristics of a rectangle, a rhombus, and a square.

3. Explain the difference between a square and a rectangle.

**ALGEBRA** In rhombus $ABCD$, $AB = 2x + 3$ and $BC = 5x$.

4. Find $x$.

5. Find $AD$.

6. Find $m\angle AEB$.

7. Find $m\angle BCD$ if $m\angle ABC = 83.2^\circ$.

**COORDINATE GEOMETRY** Given each set of vertices, determine whether $\square MNPQ$ is a rhombus, a rectangle, or a square. List all that apply. Explain your reasoning.

8. $M(0, 3), N(-3, 0), P(0, -3), Q(3, 0)$

9. $M(-4, 0), N(-3, 3), P(2, 2), Q(1, -1)$

10. PROOF Write a two-column proof.

   **Given:** $\triangle KGH, \triangle HKJ, \triangle GHIJ$, and $\triangle JKG$ are isosceles.

   **Prove:** $GHJK$ is a rhombus.

**Application** 11. REMODELING The Steiner family is remodeling their kitchen. Each side of the floor measures 10 feet. What other measurements should be made to determine whether the floor is a square?

---

**Practice and Apply**

In rhombus $ABCD$, $m\angle DAB = 2m\angle ADC$ and $CB = 6$.

12. Find $m\angle ACD$.

13. Find $m\angle DAB$.

14. Find $DA$.

15. Find $m\angle ADB$.

**ALGEBRA** Use rhombus $XYZW$ with $m\angle WYZ = 53^\circ$, $VW = 3$, $XV = 2a - 2$, and $ZV = \frac{5a + 1}{4}$.

16. Find $m\angle YZV$.

17. Find $m\angle XYW$.

18. Find $XZ$.

19. Find $XW$.

**COORDINATE GEOMETRY** Given each set of vertices, determine whether $\square EFGH$ is a rhombus, a rectangle, or a square. List all that apply. Explain your reasoning.

20. $E(1, 10), F(-4, 0), G(7, 2), H(12, 12)$

21. $E(-7, 3), F(-2, 3), G(1, 7), H(-4, 7)$

22. $E(1, 5), F(6, 5), G(6, 10), H(1, 10)$

23. $E(-2, -1), F(-4, 3), G(1, 5), H(3, 1)$
Construct each figure using a compass and ruler.

24. a square with one side 3 centimeters long
25. a square with a diagonal 5 centimeters long

Use the Venn diagram to determine whether each statement is always, sometimes, or never true.

26. A parallelogram is a square.
27. A square is a rhombus.
28. A rectangle is a parallelogram.
29. A rhombus is a rectangle.
30. A rhombus is a square.
31. A square is a rectangle.

32. DESIGN Otto Prutscher designed the plant stand at the left in 1903. The base is a square, and the base of each of the five boxes is also a square. Suppose each smaller box is one half as wide as the base. Use the information at the left to find the dimensions of the base of one of the smaller boxes.

33. PERIMETER The diagonals of a rhombus are 12 centimeters and 16 centimeters long. Find the perimeter of the rhombus.

34. ART This piece of art is Dorthea Rockburne’s *Egyptian Painting: Scribe*. The diagram shows three of the shapes shown in the piece. Use a ruler or a protractor to determine which type of quadrilateral is represented by each figure.

35. Theorem 8.16
36. Theorem 8.17

SQUASH For Exercises 37 and 38, use the diagram of the court for squash, a game similar to racquetball and tennis.

37. The diagram labels the diagonal as 11,665 millimeters. Is this correct? Explain.

38. The service boxes are squares. Find the length of the diagonal.
39. **FLAGS** Study the flags shown below. Use a ruler and protractor to determine if any of the flags contain parallelograms, rectangles, rhombi, or squares.

![Flags](image)

Denmark  | St. Vincent and The Grenadines  | Trinidad and Tobago

PROOF Write a two-column proof.

40. **Given:** \(\triangle WZY \cong \triangle WXY, \triangle WZY\)
and \(\triangle XYZ\) are isosceles.

**Prove:** \(WXYZ\) is a rhombus.

![Proof Diagram](image)

41. **Given:** \(\triangle TPX \cong \triangle QPX \cong \triangle QRX \cong \triangle TRX\)

**Prove:** \(TPQR\) is a rhombus.

42. **Given:** \(\triangle LGK \cong \triangle MJK\)

**Prove:** \(GHJK\) is a parallelogram.

43. **Given:** \(QRST\) and \(QRTV\) are rhombi.

**Prove:** \(\triangle QRT\) is equilateral.

44. **CRITICAL THINKING**
The pattern at the right is a series of rhombi that continue to form a hexagon that increases in size. Copy and complete the table.

<table>
<thead>
<tr>
<th>Hexagon</th>
<th>Number of rhombi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>(x)</td>
<td></td>
</tr>
</tbody>
</table>

45. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

**How can you ride a bicycle with square wheels?**

Include the following in your answer:
- difference between squares and rhombi, and
- how nonsquare rhombus-shaped wheels would work with the curved road.
46. Points $A$, $B$, $C$, and $D$ are on a square. The area of the square is 36 square units. Which of the following statements is true?
   - A. The perimeter of rectangle $ABCD$ is greater than 24 units.
   - B. The perimeter of rectangle $ABCD$ is less than 24 units.
   - C. The perimeter of rectangle $ABCD$ is equal to 24 units.
   - D. The perimeter of rectangle $ABCD$ cannot be determined from the information given.

47. **ALGEBRA** For all integers $x \neq 2$, let $<x> = \frac{1 + x}{x - 2}$. Which of the following has the greatest value?
   - A. $<0>$
   - B. $<1>$
   - C. $<3>$
   - D. $<4>$

**Maintain Your Skills**

**Mixed Review**

**ALGEBRA** Use rectangle $LMNP$, parallelogram $LKMJ$, and the given information to solve each problem. **(Lesson 8-4)**

48. If $LN = 10$, $LJ = 2x + 1$, and $PJ = 3x - 1$, find $x$.
49. If $m\angle PLK = 110$, find $m\angle LKM$.
50. If $m\angle MJK = 35$, find $m\angle MPN$.
51. If $MK = 6x$, $KL = 3x + 2y$, and $JN = 14 - x$, find $x$ and $y$.
52. If $m\angle LMP = m\angle PMN$, find $m\angle PJL$.

**COORDINATE GEOMETRY** Determine whether the points are the vertices of a parallelogram. Use the method indicated. **(Lesson 8-3)**

53. $P(0, 2), Q(6, 4), R(4, 0), S(-2, -2)$; Distance Formula
54. $F(1, -1), G(-4, 1), H(-3, 4), J(2, 1)$; Distance Formula
55. $K(-3, -7), L(3, 2), M(1, 7), N(-3, 1)$; Slope Formula
56. $A(-4, -1), B(-2, -5), C(1, 7), D(3, 3)$; Slope Formula

Refer to $\triangle PQS$. **(Lesson 6-4)**

57. If $RT = 16$, $QP = 24$, and $ST = 9$, find $PS$.
58. If $PT = y - 3$, $PS = y + 2$, $RS = 12$, and $QS = 16$, solve for $y$.
59. If $RT = 15$, $QP = 21$, and $PT = 8$, find $TS$.

Refer to the figure. **(Lesson 4-6)**

60. If $\overline{AG} \cong \overline{AC}$, name two congruent angles.
61. If $\overline{AJ} \cong \overline{AH}$, name two congruent angles.
62. If $\angle AFD \cong \angle ADF$, name two congruent segments.
63. If $\angle AKB \cong \angle ABK$, name two congruent segments.

**Getting Ready for the Next Lesson**

**PREREQUISITE SKILL** Solve each equation. **(To review solving equations, see pages 737 and 738.)**

64. $\frac{1}{2}(8x - 6x - 7) = 5$
65. $\frac{1}{2}(7x + 3x + 1) = 12.5$
66. $\frac{1}{2}(4x + 6 + 2x + 13) = 15.5$
67. $\frac{1}{2}(7x - 2 + 3x + 3) = 25.5$

www.geometryonline.com/self_check_quiz/sol
Kites

A kite is a quadrilateral with exactly two distinct pairs of adjacent congruent sides. In kite $ABCD$, diagonal $BD$ separates the kite into two congruent triangles. Diagonal $AC$ separates the kite into two noncongruent isosceles triangles.

Activity

Construct a kite $QRST$.

1. Draw $RT$.

2. Choose a compass setting greater than $\frac{1}{2} RT$. Place the compass at point $R$ and draw an arc above $RT$. Then without changing the compass setting, move the compass to point $T$ and draw an arc that intersects the first one. Label the intersection point $Q$. Increase the compass setting. Place the compass at $R$ and draw an arc below $RT$. Then, without changing the compass setting, draw an arc from point $T$ to intersect the other arc. Label the intersection point $S$.

3. Draw $QRST$.

Model

1. Draw $QS$ in kite $QRST$. Use a protractor to measure the angles formed by the intersection of $QS$ and $RT$.

2. Measure the interior angles of kite $QRST$. Are any congruent?

3. Label the intersection of $QS$ and $RT$ as point $N$. Find the lengths of $QN$, $NS$, $TN$, and $NR$. How are they related?

4. How many pairs of congruent triangles can be found in kite $QRST$?


Analyze

6. Use your observations and measurements of kites $QRST$ and $JKLM$ to make conjectures about the angles, sides, and diagonals of kites.
8-6 Trapezoids

**What You’ll Learn**
- Recognize and apply the properties of trapezoids.
- Solve problems involving the medians of trapezoids.

**Vocabulary**
- trapezoid
- isosceles trapezoid
- median

**How are trapezoids used in architecture?**
The Washington Monument in Washington, D.C., is an obelisk made of white marble. The width of the base is longer than the width at the top. Each face of the monument is an example of a trapezoid.

**Properties of Trapezoids**
A trapezoid is a quadrilateral with exactly one pair of parallel sides. The parallel sides are called bases. The base angles are formed by a base and one of the legs. The nonparallel sides are called legs.

If the legs are congruent, then the trapezoid is an isosceles trapezoid. Theorems 8.18 and 8.19 describe two characteristics of isosceles trapezoids.

**Theorems**

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.18</td>
<td>Both pairs of base angles of an isosceles trapezoid are congruent.</td>
<td>$\triangle DAB \cong \triangle CBA$ $\triangle ADC \cong \triangle BCD$ $\overline{AC} \cong \overline{BD}$</td>
</tr>
<tr>
<td>8.19</td>
<td>The diagonals of an isosceles trapezoid are congruent.</td>
<td></td>
</tr>
</tbody>
</table>

**Example 1**

**Proof of Theorem 8.19**
Write a flow proof of Theorem 8.19.

**Given:** $MNOP$ is an isosceles trapezoid.

**Prove:** $MO \cong NO$ $\overline{MN} \parallel \overline{OP}$ $\measuredangle MPO = \measuredangle NOP$ $\overline{MPO} = \overline{NOP}$ $\measuredangle MPO = \measuredangle NOP$ $\triangle MPO = \triangle NOP$ $\triangle MPO \cong \triangle NOP$ $MO \cong NO$ $\triangle MPO \cong \triangle NOP$ $\triangle MPO \cong \triangle NOP$

**Standard G.8.a** The student will investigate and identify properties of quadrilaterals involving opposite sides ...;
**Standard G.8.b** The student will prove these properties of quadrilaterals, using algebraic ... methods as well as deductive reasoning; and
**Standard G.8.c** The student will use properties of quadrilaterals to solve practical problems.
**Example 2** Identify Isoceles Trapezoids

**ART** The sculpture pictured is *Zim Zum I* by Barnett Newman. The walls are connected at right angles. In perspective, the rectangular panels appear to be trapezoids. Use a ruler and protractor to determine if the images of the front panels are isosceles trapezoids. Explain.

The panel on the left is an isosceles trapezoid. The bases are parallel and are different lengths. The legs are not parallel and they are the same length.

The panel on the right is not an isosceles trapezoid. Each side is a different length.

**Example 3** Identify Trapezoids

**COORDINATE GEOMETRY** \(JKLM\) is a quadrilateral with vertices \(J(-18, -1)\), \(K(-6, 8)\), \(L(18, 1)\), and \(M(-18, -26)\).

a. Verify that \(JKLM\) is a trapezoid.

A quadrilateral is a trapezoid if exactly one pair of opposite sides are parallel. Use the Slope Formula.

\[
\text{slope of } JK = \frac{-1 - 8}{-18 - (-6)} = \frac{-9}{12} \text{ or } \frac{3}{4} \\
\text{slope of } ML = \frac{1 - (-26)}{18 - (-18)} = \frac{27}{36} \text{ or } \frac{3}{4} \\
\text{slope of } JM = \frac{-1 - (-26)}{-18 - (-18)} = \frac{25}{0} \text{ or undefined} \\
\text{slope of } KL = \frac{1 - 8}{18 - (-6)} = \frac{-7}{24}
\]

Exactly one pair of opposite sides are parallel, \(JK\) and \(ML\). So, \(JKLM\) is a trapezoid.

b. Determine whether \(JKLM\) is an isosceles trapezoid. Explain.

First use the Distance Formula to show that the legs are congruent.

\[
JM = \sqrt{[-18 - (-18)]^2 + [-1 - (-26)]^2} = \sqrt{625} \text{ or } 25 \\
KL = \sqrt{(-6 - 18)^2 + (8 - 1)^2} = \sqrt{576 + 49} = \sqrt{625} \text{ or } 25
\]

Since the legs are congruent, \(JKLM\) is an isosceles trapezoid.

**MEDIANs OF TRAPEZOIDS** The segment that joins midpoints of the legs of a trapezoid is the **median**.

The median of a trapezoid can also be called a **midsegment**. Recall from Lesson 6-4 that the midsegment of a triangle is the segment joining the midpoints of two sides. The median of a trapezoid has the same properties as the midsegment of a triangle. You can construct the median of a trapezoid using a compass and a straightedge.
Theorem 8.20

The median of a trapezoid is parallel to the bases, and its measure is one-half the sum of the measures of the bases.

Example: \( EF = \frac{1}{2} (AB + DC) \)

**Example 4** Median of a Trapezoid

**ALGEBRA**  
QRST is an isosceles trapezoid with median \( XY \).

a. Find \( TS \) if \( QR = 22 \) and \( XY = 15 \).

\[
XY = \frac{1}{2}(QR + TS) \quad \text{Theorem 8.20}
\]
\[
15 = \frac{1}{2}(22 + TS) \quad \text{Substitution}
\]
\[
30 = 22 + TS \quad \text{Multiply each side by 2.}
\]
\[
8 = TS \quad \text{Subtract 22 from each side.}
\]

b. Find \( m\angle 1, m\angle 2, m\angle 3, \) and \( m\angle 4 \) if \( m\angle 1 = 4a - 10 \) and \( m\angle 3 = 3a + 32.5 \).

Since \( QR \parallel TS \), \( \angle 1 \) and \( \angle 3 \) are supplementary. Because this is an isosceles trapezoid, \( \angle 1 \equiv \angle 2 \) and \( \angle 3 \equiv \angle 4 \).

\[
m\angle 1 + m\angle 3 = 180 \quad \text{Consecutive Interior Angles Theorem}
\]
\[
4a - 10 + 3a + 32.5 = 180 \quad \text{Substitution}
\]
\[
7a + 22.5 = 180 \quad \text{Combine like terms.}
\]
\[
7a = 157.5 \quad \text{Subtract 22.5 from each side.}
\]
\[
a = 22.5 \quad \text{Divide each side by 7.}
\]

If \( a = 22.5 \), then \( m\angle 1 = 80 \) and \( m\angle 3 = 100 \).

Because \( \angle 1 \equiv \angle 2 \) and \( \angle 3 \equiv \angle 4 \), \( m\angle 2 = 80 \) and \( m\angle 4 = 100 \).
**Check for Understanding**

1. **Concept Check** List the minimum requirements to show that a quadrilateral is a trapezoid.
2. **Guided Practice** Make a chart comparing the characteristics of the diagonals of a trapezoid, a rectangle, a square, and a rhombus. (Hint: Use the types of quadrilaterals as column headings and the properties of diagonals as row headings.)
3. **Application** Draw an isosceles trapezoid and a trapezoid that is not isosceles. Draw the median for each. Is the median parallel to the bases in both trapezoids?

**Practice and Apply**

**COORDINATE GEOMETRY** 
QRST is a quadrilateral with vertices Q(−3, 2), R(−1, 6), S(4, 6), T(6, 2).
4. Verify that QRST is a trapezoid.
5. Determine whether QRST is an isosceles trapezoid. Explain.

6. **Proof** CDFG is an isosceles trapezoid with bases CD and FG. Write a flow proof to prove ∠DGF ≅ ∠CFG.

7. **Algebra** EFGH is an isosceles trapezoid with bases EF and GH and median YZ. If EF = 3x + 8, HG = 4x − 10, and YZ = 13, find x.

**Extra Practice** See page 770.

**Homework Help**

- Examples: 3, 4, 2, 1

---

**Application**

8. **Photography** Photographs can show a building in a perspective that makes it appear to be a different shape. Identify the types of quadrilaterals in the photograph.

**COORDINATE GEOMETRY** 
For each quadrilateral whose vertices are given, 
a. verify that the quadrilateral is a trapezoid, and 
b. determine whether the figure is an isosceles trapezoid.
9. A(−3, 3), B(−4, −1), C(5, −1), D(2, 3)
10. G(−5, −4), H(5, 4), J(0, 5), K(−5, 1)
11. C(−1, 1), D(−5, −3), E(−4, −10), F(6, 0)
12. Q(−12, 1), R(−9, 4), S(−4, 3), T(−11, −4)

**Algebra** Find the missing measure(s) for the given trapezoid.

13. For trapezoid DEGH, X and Y are midpoints of the legs. Find DE.

14. For trapezoid RSTV, A and B are midpoints of the legs. Find VT.
15. For isosceles trapezoid $WXYZ$, find the length of the median, $m \angle W$, and $m \angle Z$.

![Diagram of trapezoid WXYZ with angles labeled 70° and 20°]

16. For trapezoid $QRST$, $A$ and $B$ are midpoints of the legs. Find $AB$, $m \angle Q$, and $m \angle S$.

For Exercises 17 and 18, use trapezoid $QRST$.

17. Let $GH$ be the median of $RSBA$. Find $GH$.

18. Let $JK$ be the median of $ABTQ$. Find $JK$.

19. **ALGEBRA** $JKLM$ is a trapezoid with $JK \parallel LM$ and median $RP$. Find $RP$ if $JK = 2(x + 3)$, $RP = 5 + x$, and $ML = \frac{1}{2}x - 1$.

20. **DESIGN** The bagua is a tool used in Feng Shui design. This bagua consists of two regular octagons centered around a yin-yang symbol. How can you determine the type of quadrilaterals in the bagua?

21. **SEWING** Madison is making a valance for a window treatment. She is using striped fabric cut on the bias, or diagonal, to create a chevron pattern. Identify the polygons formed in the fabric.

**COORDINATE GEOMETRY** Determine whether each figure is a trapezoid, a parallelogram, a square, a rhombus, or a quadrilateral. Choose the most specific term. Explain.

22. [Diagram of a parallelogram with vertices $A(1, 2)$, $B(4, 4)$, $C(5, 1)$, and $D(2, -1)$]

23. [Diagram of a trapezoid with vertices $G(-2, 2)$, $H(4, 2)$, $J(6, -1)$, and $K(-4, -1)$]

24. [Diagram of a trapezoid with vertices $M(-3, 1)$, $N(1, 3)$, $O(3, -1)$, and $P(-2, -2)$]

25. [Diagram of a trapezoid with vertices $Q(-3, 0)$, $R(0, 3)$, $S(3, 0)$, and $T(0, -3)$]

**COORDINATE GEOMETRY** For Exercises 26–28, refer to quadrilateral $PQRS$.

26. Determine whether the figure is a trapezoid. If so, is it isosceles? Explain.

27. Find the coordinates of the midpoints of $PQ$ and $RS$, and label them $A$ and $B$.

28. Find $AB$ without using the Distance Formula.
COORDINATE GEOMETRY  For Exercises 29–31, refer to quadrilateral DEFG.

29. Determine whether the figure is a trapezoid. If so, is it isosceles? Explain.

30. Find the coordinates of the midpoints of DE and GF, and label them W and V.

31. Find WV without using the Distance Formula.

32. Given: HGK, δHJK ≅ δJHK, HG ∥ JK
   Prove: GHJK is an isosceles trapezoid.

33. Given: ΔTZX ≅ ΔYXZ, WX ∥ Zy
   Prove: XYZW is a trapezoid.

34. Given: ZYXP is an isosceles trapezoid.
   Prove: ΔPWX is isosceles.

35. Given: E and C are midpoints of AD and DB. AD ≅ DB
   Prove: ABCE is an isosceles trapezoid.

36. Write a paragraph proof of Theorem 8.18.

CONSTRUCTION  Use a compass and ruler to construct each figure.

37. an isosceles trapezoid

38. trapezoid with a median 2 centimeters long

39. CRITICAL THINKING  In RSTV, RS = 6, VT = 3, and RX is twice the length of XV. Find XY.

40. CRITICAL THINKING  Is it possible for an isosceles trapezoid to have two right base angles? Explain.

41. WRITING IN MATH  Answer the question that was posed at the beginning of the lesson.

How are trapezoids used in architecture?
Include the following in your answer:
• the characteristics of a trapezoid, and
• other examples of trapezoids in architecture.

42. SHORT RESPONSE  What type of quadrilateral is WXYZ?
Justify your answer.
43. **ALGEBRA** In the figure, which point lies within the shaded region?

   - A (−2, 4)
   - B (−1, 3)
   - C (1, −3)
   - D (2, −4)

44. **COORDINATE GEOMETRY** For Exercises 49–51, refer to quadrilateral RSTV. (Lesson 8-4)

49. Find RS and TV.

50. Find the coordinates of the midpoints of RT and SV.

51. Is RSTV a rectangle? Explain.

52. Solve each proportion. (Lesson 6-1)

   - \[ \frac{16}{38} = \frac{24}{y} \]
   - \[ \frac{y}{6} = \frac{17}{30} \]
   - \[ \frac{y + 4}{5} = \frac{20}{28} \]
   - \[ \frac{2y}{9} = \frac{52}{36} \]

53. **PREREQUISITE SKILL** Write an expression for the slope of the segment given the coordinates of the endpoints. (To review slope, see Lesson 3-3.)

   - 56. \( (0, a), (-a, 2a) \)
   - 57. \( (-a, b), (a, b) \)
   - 58. \( (c, c), (c, d) \)
   - 59. \( (a, -b), (2a, b) \)
   - 60. \( (3a, 2b), (b, -a) \)
   - 61. \( (b, c), (-b, -c) \)

**Practice Quiz 2**

Quadrilateral ABCD is a rectangle. (Lesson 8-4)

1. Find \( x \).

2. Find \( y \).

3. **COORDINATE GEOMETRY** Determine whether MNPQ is a rhombus, a rectangle, or a square for \( M(-5, -3), N(-2, 3), P(1, -3), \) and \( Q(-2, -9) \). List all that apply. Explain. (Lesson 8-5)

For trapezoid TRSV, M and N are midpoints of the legs. (Lesson 8-6)

4. If VS = 21 and TR = 44, find MN.

5. If TR = 32 and MN = 25, find VS.
**Hierarchy of Polygons**

A *hierarchy* is a ranking of classes or sets of things. Examples of some classes of polygons are rectangles, rhombi, trapezoids, parallelograms, squares, and quadrilaterals. These classes are arranged in the hierarchy below.

Use the following information to help read the hierarchy diagram.

- The class that is the broadest is listed first, followed by the other classes in order. For example, *polygons* is the broadest class in the hierarchy diagram above, and *squares* is a very specific class.
- Each class is contained within any class linked above it in the hierarchy. For example, *all squares* are also rhombi, rectangles, parallelograms, quadrilaterals, and polygons. However, an isosceles trapezoid is not a square or a kite.
- Some, but not all, elements of each class are contained within lower classes in the hierarchy. For example, some trapezoids are isosceles trapezoids, and some rectangles are squares.

**Reading to Learn**

Refer to the hierarchy diagram at the right. Write *true*, *false*, or *not enough information* for each statement.

1. All mogs are jums.
2. Some jebs are jums.
3. All lems are jums.
4. Some wibs are jums.
5. All mogs are bips.
6. Draw a hierarchy diagram to show these classes: equilateral triangles, polygons, isosceles triangles, triangles, and scalene triangles.
The first step to using a coordinate proof is to place the figure on the coordinate plane. The placement of the figure can simplify the steps of the proof.

Some examples of quadrilaterals placed on the coordinate plane are given below. Notice how the figures have been placed so the coordinates of the vertices are as simple as possible.

Some examples of quadrilaterals placed on the coordinate plane are given below. Notice how the figures have been placed so the coordinates of the vertices are as simple as possible.

**Positioning a Square**

Position and label a square with sides $a$ units long on the coordinate plane.

- Let $A$, $B$, $C$, and $D$ be vertices of a square with sides $a$ units long.
- Place the square with vertex $A$ at the origin, $AB$ along the positive $x$-axis, and $AD$ along the $y$-axis. Label the vertices $A$, $B$, $C$, and $D$.
- The $y$-coordinate of $B$ is 0 because the vertex is on the $x$-axis. Since the side length is $a$, the $x$-coordinate is $a$.
- $D$ is on the $y$-axis so the $x$-coordinate is 0. The $y$-coordinate is $0 + a$ or $a$.
- The $x$-coordinate of $C$ is also $a$. The $y$-coordinate is $0 + a$ or $a$ because the side $BC$ is $a$ units long.

Some examples of quadrilaterals placed on the coordinate plane are given below. Notice how the figures have been placed so the coordinates of the vertices are as simple as possible.

- **Rectangle**
- **Parallelogram**
- **Isosceles Trapezoid**
- **Rhombus**
Once a figure has been placed on the coordinate plane, we can prove theorems using the Slope, Midpoint, and Distance Formulas.

Example 2 Find Missing Coordinates

Name the missing coordinates for the parallelogram. Opposite sides of a parallelogram are congruent and parallel. So, the $y$-coordinate of $D$ is $a$.

The length of $AB$ is $b$, and the length of $DC$ is $b$. So, the $x$-coordinate of $D$ is $(b + c) - b$ or $c$. The coordinates of $D$ are $(c, a)$.

PROVE THEOREMS Once a figure has been placed on the coordinate plane, we can prove theorems using the Slope, Midpoint, and Distance Formulas.

Geometry Software Investigation

Quadrilaterals

Model

- Use The Geometer’s Sketchpad to draw a quadrilateral $ABCD$ with no two sides parallel or congruent.
- Construct the midpoints of each side.
- Draw the quadrilateral formed by the midpoints of the segments.

Analyze

1. Measure each side of the quadrilateral determined by the midpoints of $ABCD$.
2. What type of quadrilateral is formed by the midpoints? Justify your answer.

In this activity, you discover that the quadrilateral formed from the midpoints of any quadrilateral is a parallelogram. You will prove this in Exercise 22.

Example 3 Coordinate Proof

Place a square on a coordinate plane. Label the midpoints of the sides, $M, N, P$, and $Q$. Write a coordinate proof to prove that $MNPQ$ is a square.

The first step is to position a square on the coordinate plane. Label the vertices to make computations as simple as possible.

Given: $ABCD$ is a square. $M, N, P$, and $Q$ are midpoints.

Prove: $MNPQ$ is a square.

Proof:

By the Midpoint Formula, the coordinates of $M, N, P$, and $Q$ are as follows.

$M\left(\frac{2a}{2}, \frac{0 + 0}{2}\right) = (a, 0)$

$P\left(\frac{0 + 2a}{2}, \frac{2a + 2a}{2}\right) = (a, 2a)$

$N\left(\frac{2a + 2a}{2}, \frac{2a + 0}{2}\right) = (2a, a)$

$Q\left(\frac{0 + 0}{2}, \frac{0 + 2a}{2}\right) = (0, a)$
Find the slopes of $\overline{QP}$, $\overline{MN}$, $\overline{QM}$, and $\overline{PN}$.

- slope of $\overline{QP} = \frac{2a - a}{a - 0} = 1$
- slope of $\overline{MN} = \frac{a - 0}{2a - a} = 1$
- slope of $\overline{QM} = \frac{a - 0}{0 - a} = -1$
- slope of $\overline{PN} = \frac{2a - a}{a - 2a} = -1$

Each pair of opposite sides is parallel, so they have the same slope. Consecutive sides form right angles because their slopes are negative reciprocals.

Use the Distance Formula to find the length of $\overline{QP}$ and $\overline{QM}$.

\[
\overline{QP} = \sqrt{(a - 0)^2 + (2a - a)^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}
\]

\[
\overline{QM} = \sqrt{(a - 0)^2 + (0 - a)^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}
\]

$MNPQ$ is a square because each pair of opposite sides is parallel, and consecutive sides form right angles and are congruent.

**Example 4 Properties of Quadrilaterals**

**PARKING** Write a coordinate proof to prove that the sides of the parking space are parallel.

**Given:** $14x - 6y = 0; \quad 7x - 3y = 56$

**Prove:** $A\parallel D \parallel B\parallel C$

**Proof:** Rewrite both equations in slope-intercept form.

\[
\begin{align*}
14x - 6y &= 0 \\
-6y &= -14x \\
y &= \frac{7}{3}x
\end{align*}
\]

\[
\begin{align*}
7x - 3y &= 56 \\
-3y &= -7x + 56 \\
y &= \frac{7}{3}x - \frac{56}{3}
\end{align*}
\]

Since $AD$ and $BC$ have the same slope, they are parallel.

**Check for Understanding**

**Concept Check**

1. Explain how to position a quadrilateral to simplify the steps of the proof.

2. OPEN ENDED Position and label a trapezoid with two vertices on the $y$-axis.

**Guided Practice**

Position and label the quadrilateral on the coordinate plane.

3. rectangle with length $a$ units and height $a + b$ units

Name the missing coordinates for each quadrilateral.

4.

5.

Write a coordinate proof for each statement.

6. The diagonals of a parallelogram bisect each other.

7. The diagonals of a square are perpendicular.
8. **STATES** The state of Tennessee can be separated into two shapes that resemble quadrilaterals. Write a coordinate proof to prove that $DEFG$ is a trapezoid. All measures are approximate and given in kilometers.

![Tennessee map](image)

**Application**

**Practice and Apply**

Position and label each quadrilateral on the coordinate plane.

9. isosceles trapezoid with height $c$ units, bases $a$ units and $a + 2b$ units

10. parallelogram with side length $c$ units and height $b$ units

Name the missing coordinates for each parallelogram or trapezoid.

11. 

12. 

13. 

14. 

15. 

16. 

Position and label each figure on the coordinate plane. Then write a coordinate proof for each of the following.

17. The diagonals of a rectangle are congruent.

18. If the diagonals of a parallelogram are congruent, then it is a rectangle.

19. The diagonals of an isosceles trapezoid are congruent.

20. The median of an isosceles trapezoid is parallel to the bases.

21. The segments joining the midpoints of the sides of a rectangle form a rhombus.

22. The segments joining the midpoints of the sides of a quadrilateral form a parallelogram.

23. **CRITICAL THINKING** $A$ has coordinates $(0, 0)$, and $B$ has coordinates $(a, b)$. Find the coordinates of $C$ and $D$ so $ABCD$ is an isosceles trapezoid.
**ARCHITECTURE**  For Exercises 24–26, use the following information.
The Leaning Tower of Pisa is approximately 60 meters tall, from base to belfry. The tower leans about 5.5° so the top level is 4.5 meters over the first level.

24. Position and label the tower on a coordinate plane.
25. Is it possible to write a coordinate proof to prove that the sides of the tower are parallel? Explain.
26. From the given information, what conclusion can be drawn?

27. **WRITING IN MATH**  Answer the question that was posed at the beginning of the lesson.

**How is the coordinate plane used in proofs?**
Include the following in your answer:
• guidelines for placing a figure on a coordinate grid, and
• an example of a theorem from this chapter that could be proved using the coordinate plane.

28. In the figure, $ABCD$ is a parallelogram. What are the coordinates of point $D$?
   \[ \text{A} (a, c + b) \quad \text{B} (c + b, a) \quad \text{C} (b - c, a) \quad \text{D} (c - b, a) \]

29. **ALGEBRA**  If $p = -5$, then $5 - p^2 - p = \ ?$.
   \[ \text{A} \ -15 \quad \text{B} \ -5 \quad \text{C} \ 10 \quad \text{D} \ 30 \]

---

**Maintain Your Skills**

**Mixed Review**
30. **PROOF**  Write a two-column proof.  *(Lesson 8-6)*

   **Given:**  $MNOP$ is a trapezoid with bases $MN$ and $OP$.  $MN \cong QO$

   **Prove:**  $MNOQ$ is a parallelogram.

   $JKLM$ is a rectangle.  $MLPR$ is a rhombus.  $\angle JMK \cong \angle RMP$,  $m\angle JMK = 55$, and $m\angle MRP = 70$.  *(Lesson 8-5)*

31. Find $m\angle MPR$.
32. Find $m\angle KML$.
33. Find $m\angle KLP$.

Find the geometric mean between each pair of numbers.  *(Lesson 7-1)*
34. 7 and 14
35. $2\sqrt{5}$ and $6\sqrt{5}$

Write an expression relating the given pair of angle measures.  *(Lesson 5-5)*
36. $m\angle WVX$, $m\angle VXY$
37. $m\angle XZV$, $m\angle VXZ$
38. $m\angle XYV$, $m\angle VXY$
39. $m\angle XYZ$, $m\angle ZXY$
Vocabulary and Concept Check

diagonal (p. 404)  median (p. 440)  rhombus (p. 431)
isosceles trapezoid (p. 439)  parallelogram (p. 411)  square (p. 432)
kite (p. 438)  rectangle (p. 424)  trapezoid (p. 439)

A complete list of postulates and theorems can be found on pages R1–R8.

Exercises  State whether each sentence is true or false. If false, replace the underlined term to make a true sentence.
1. The diagonals of a rhombus are perpendicular.
2. All squares are rectangles.
3. If a parallelogram is a rhombus, then the diagonals are congruent.
4. Every parallelogram is a quadrilateral.
5. A(n) rhombus is a quadrilateral with exactly one pair of parallel sides.
6. Each diagonal of a rectangle bisects a pair of opposite angles.
7. If a quadrilateral is both a rhombus and a rectangle, then it is a square.
8. Both pairs of base angles in a(n) isosceles trapezoid are congruent.

Lesson-by-Lesson Review

8-1  Angles of Polygons

See pages 404–409.

Concept Summary
- If a convex polygon has \( n \) sides and the sum of the measures of its interior angles is \( S \), then \( S = 180(n - 2) \).
- The sum of the measures of the exterior angles of a convex polygon is 360.

Example
Find the measure of an interior angle of a regular decagon.

\[
S = 180(n - 2) \quad \text{Interior Angle Sum Theorem}
\]
\[
= 180(10 - 2) \quad n = 10
\]
\[
= 180(8) \text{ or } 1440 \quad \text{Simplify.}
\]

The measure of each interior angle is \( 1440 \div 10 \), or 144.

Exercises  Find the measure of each interior angle of a regular polygon given the number of sides. See Example 1 on page 405.

9. 6  10. 15  11. 4  12. 20

ALGEBRA  Find the measure of each interior angle. See Example 3 on page 405.

13. \[
\begin{align*}
X & = a \\
Y & = (a - 28) \\
Z & = (a + 2)
\end{align*}
\]

14. \[
\begin{align*}
B & = (1.5x + 3) \\
D & = (2x - 22) \\
E & = (x + 27)
\end{align*}
\]
Parallelograms

Concept Summary
- In a parallelogram, opposite sides are parallel and congruent, opposite angles are congruent, and consecutive angles are supplementary.
- The diagonals of a parallelogram bisect each other.

Example

WXYZ is a parallelogram. Find \( m\angle YZW \) and \( m\angle XWZ \).

\[
\begin{align*}
m\angle YZW &= m\angle WXY & \text{Opp. \ of \ } \square \text{ are } \cong, \\
m\angle YZW &= 82 + 33 \text{ or } 115 & m\angle WXY = m\angle WZ + m\angle YX \\
m\angle XWZ + m\angle WXY &= 180 & \text{Cons. \ } \triangle \text{ in } \square \text{ are suppl.} \\
m\angle XWZ + (82 + 33) &= 180 & m\angle WXY = m\angle WZ + m\angle YX \\
m\angle XWZ + 115 &= 180 & \text{Simplify.} \\
m\angle XWZ &= 65 & \text{Subtract 115 from each side.}
\end{align*}
\]

Exercises
Use \( \square ABCD \) to find each measure.

15. \( m\angle BCD \) 
16. \( AF \)
17. \( m\angle BDC \) 
18. \( BC \)
19. \( CD \) 
20. \( m\angle ADC \)

Tests for Parallelograms

Concept Summary
A quadrilateral is a parallelogram if any one of the following is true.
- Both pairs of opposite sides are parallel and congruent.
- Both pairs of opposite angles are congruent.
- Diagonals bisect each other.
- A pair of opposite sides is both parallel and congruent.

Example

COORDINATE GEOMETRY Determine whether the figure with vertices \( A(-5, 3), B(-1, 5), C(6, 1), \) and \( D(2, -1) \) is a parallelogram. Use the Distance and Slope Formulas.

\[
\begin{align*}
AB &= \sqrt{(-5 - (-1))^2 + (3 - 5)^2} \\
&= \sqrt{(-4)^2 + (-2)^2} \text{ or } \sqrt{20} \\
CD &= \sqrt{(6 - 2)^2 + [1 - (-1)]^2} \\
&= \sqrt{4^2 + 2^2} \text{ or } \sqrt{20} \\
\text{Since } AB &= CD, AB \cong CD. \\
\text{slope of } AB &= \frac{5 - 3}{-1 - (-5)} \text{ or } \frac{1}{2} & \text{slope of } CD &= \frac{-1 - 1}{2 - 6} \text{ or } \frac{1}{2}
\end{align*}
\]

\( AB \) and \( CD \) have the same slope, so they are parallel. Since one pair of opposite sides is congruent and parallel, \( ABCD \) is a parallelogram.
Exercises  Determine whether the figure with the given vertices is a parallelogram. Use the method indicated.  See Example 5 on page 420.
21.  \( A(-2, 5), B(4, 4), C(6, -3), D(-1, -2) \); Distance Formula
22.  \( H(0, 4), J(-4, 6), K(5, 6), L(9, 4) \); Midpoint Formula
23.  \( S(-2, -1), T(2, 5), V(-10, 13), W(-14, 7) \); Slope Formula

Rectangles

Concept Summary
- A rectangle is a quadrilateral with four right angles and congruent diagonals.
- If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Example

Quadrilateral \( KLMN \) is a rectangle.
If \( PL = x^2 - 1 \) and \( PM = 4x + 11 \), find \( x \).
The diagonals of a rectangle are congruent and bisect each other, so \( PL \equiv PM \).

\[
\begin{align*}
PL & \equiv PM \\
\text{Diag. are } & \equiv \text{ and bisect each other.} \\
PL & = PM \\
\text{Def. of } & \equiv \text{ angles} \\
x^2 - 1 & = 4x + 11 \\
\text{Subtraction} \\
x^2 - 1 - 4x & = 11 \\
\text{Subtract 4x from each side.} \\
x^2 - 4x & = 12 \\
\text{Subtract 11 from each side.} \\
(x + 2)(x - 6) & = 0 \\
\text{Factor.} \\
x + 2 & = 0 \quad x - 6 = 0 \\
x & = -2 \quad x = 6 \\
\text{The value of } x & \text{ is -2 or 6.}
\end{align*}
\]

Exercises  \( ABCD \) is a rectangle.  See Examples 1 and 2 on pages 425 and 426.
24.  If \( AC = 9x - 1 \) and \( AF = 2x + 7 \), find \( AF \).
25.  If \( m\angle 1 = 12x + 4 \) and \( m\angle 2 = 16x - 12 \), find \( m\angle 2 \).
26.  If \( CF = 4x + 1 \) and \( DF = x + 13 \), find \( x \).
27.  If \( m\angle 2 = 70 - 4x \) and \( m\angle 5 = 18x - 8 \), find \( m\angle 5 \).

COORDINATE GEOMETRY  Determine whether \( RSTV \) is a rectangle given each set of vertices. Justify your answer.  See Example 4 on pages 426 and 427.
28.  \( R(-3, -5), S(0, -5), T(3, 4), V(0, 4) \)
29.  \( R(0, 0), S(6, 3), T(4, 7), V(-2, 4) \)
8-5 Rhombi and Squares

Concept Summary

- A rhombus is a quadrilateral with each side congruent, diagonals that are perpendicular, and each diagonal bisecting a pair of opposite angles.
- A quadrilateral that is both a rhombus and a rectangle is a square.

Example

Use rhombus $JKLM$ to find $m\angle JMK$ and $m\angle KJM$.

The opposite sides of a rhombus are parallel, so $KL \parallel JM$. $\angle JMK \cong \angle LKM$ because alternate interior angles are congruent.

$m\angle JMK = m\angle LKM$  \hspace{0.5cm} \text{Definition of congruence} \\
= 28 \hspace{0.5cm} \text{Substitution}

The diagonals of a rhombus bisect the angles, so $\angle KJM \cong \angle KLM$.

$m\angle KJM + m\angle KJM = 180$  \hspace{0.5cm} \text{Cons. \hspace{0.1cm} \triangle in \hspace{0.1cm} \square are suppl.} \\
m\angle KJM + (m\angle KJM + m\angle LKM) = 180 \hspace{0.5cm} m\angle KJM = m\angle KJM + m\angle LKM \\
m\angle KJM + (28 + 28) = 180 \hspace{0.5cm} \text{Substitution} \\
m\angle KJM + 56 = 180 \hspace{0.5cm} \text{Add.} \\
m\angle KJM = 124 \hspace{0.5cm} \text{Subtract 56 from each side.}

Exercises

Use rhombus $ABCD$ with $m\angle 1 = 2x + 20$, $m\angle 2 = 5x - 4$, $AC = 15$, and $m\angle 3 = y^2 + 26$. \text{See Example 2 on page 432.}

30. Find $x$.
31. Find $AF$.
32. Find $y$.

8-6 Trapezoids

Concept Summary

- In an isosceles trapezoid, both pairs of base angles are congruent and the diagonals are congruent.
- The median of a trapezoid is parallel to the bases, and its measure is one-half the sum of the measures of the bases.

Example

$RSTV$ is a trapezoid with bases $RV$ and $ST$ and median $MN$. Find $x$ if $MN = 60$, $ST = 4x - 1$, and $RV = 6x + 11$.

$MN = \frac{1}{2}(ST + RV)$ \\
$60 = \frac{1}{2}[(4x - 1) + (6x + 11)] \hspace{0.5cm} \text{Substitution} \\
120 = 4x - 1 + 6x + 11 \hspace{0.5cm} \text{Multiply each side by 2.} \\
120 = 10x + 10 \hspace{0.5cm} \text{Simplify.} \\
110 = 10x \hspace{0.5cm} \text{Subtract 10 from each side.} \\
11 = x \hspace{0.5cm} \text{Divide each side by 10.}$
Exercises  Find the missing value for the given trapezoid.
See Example 4 on page 441.
33. For isosceles trapezoid $ABCD$, $X$ and $Y$ are midpoints of the legs. Find $m \angle XBC$ if $m \angle ADY = 78$.

34. For trapezoid $JKLM$, $A$ and $B$ are midpoints of the legs. If $AB = 57$ and $KL = 21$, find $JM$.

Coordinate Proof with Quadrilaterals

Concept Summary
- Position a quadrilateral so that a vertex is at the origin and at least one side lies along an axis.

Example
Position and label rhombus $RSTV$ on the coordinate plane. Then write a coordinate proof to prove that each pair of opposite sides is parallel.

First, draw rhombus $RSTV$ on the coordinate plane. Label the coordinates of the vertices.

Given: $RSTV$ is a rhombus.

Prove: $RV \parallel ST$, $RS \parallel VT$

Proof:

\[ \text{slope of } RV = \frac{c-0}{b-0} \text{ or } \frac{c}{b} \]
\[ \text{slope of } ST = \frac{c-0}{(a+b)-a} \text{ or } \frac{c}{b} \]
\[ \text{slope of } RS = \frac{0-0}{a-0} \text{ or } 0 \]
\[ \text{slope of } VT = \frac{c-0}{(a+b)-b} \text{ or } 0 \]

$RV$ and $ST$ have the same slope. So $RV \parallel ST$. $RS$ and $VT$ have the same slope, and $RS \parallel VT$.

Exercises  Position and label each figure on the coordinate plane. Then write a coordinate proof for each of the following. See Example 3 on pages 448 and 449.
35. The diagonals of a square are perpendicular.
36. A diagonal separates a parallelogram into two congruent triangles.

Name the missing coordinates for each quadrilateral. See Example 2 on page 448.
37. 

38.
Determine whether each conditional is true or false. If false, draw a counterexample.
1. If a quadrilateral has four right angles, then it is a rectangle.
2. If a quadrilateral has all four sides congruent, then it is a square.
3. If the diagonals of a quadrilateral are perpendicular, then it is a rhombus.

Complete each statement about \( \square \text{FGHK} \). Justify your answer.
4. \( \overline{HK} \equiv \) ___.
5. \( \angle FKI \equiv \) ___.
6. \( \angle FKH \equiv \) ___.
7. \( \overline{GH} \parallel \) ___.

Determine whether the figure with the given vertices is a parallelogram. Justify your answer.
8. \( A(4, 3), B(6, 0), C(4, -8), D(2, -5) \)
9. \( S(-2, 6), T(2, 11), V(3, 8), W(-1, 3) \)
10. \( F(7, -3), G(4, -2), H(6, 4), J(12, 2) \)
11. \( W(-4, 2), X(-3, 6), Y(2, 7), Z(1, 3) \)

**ALGEBRA**

12. If \( QP = 3x + 11 \) and \( PS = 4x + 8 \), find \( QS \).
13. If \( m\angle QTR = 2x^2 - 7 \) and \( m\angle SRT = x^2 + 18 \), find \( m\angle QTR \).

**COORDINATE GEOMETRY**

Determine whether \( \square \text{ABCD} \) is a rhombus, a rectangle, or a square. List all that apply. Explain your reasoning.
14. \( A(12, 0), B(6, -6), C(0, 0), D(6, 6) \)
15. \( A(-2, 4), B(5, 6), C(12, 4), D(5, 2) \)

Name the missing coordinates for each quadrilateral.
16. \( N(0, c), P(? , ?) \)
17. \( D(? , ?), C(? , ?) \)

18. Position and label an isosceles trapezoid on the coordinate plane. Write a coordinate proof to prove that the median is parallel to each base.

19. **SAILING**

Many large sailboats have a keel to keep the boat stable in high winds. A keel is shaped like a trapezoid with its top and bottom parallel. If the root chord is 9.8 feet and the tip chord is 7.4 feet, find the length of the mid-chord.

20. **STANDARDIZED TEST PRACTICE**

The measure of an interior angle of a regular polygon is 108. Find the number of sides.

\[ \text{A} \ 8 \quad \text{B} \ 6 \quad \text{C} \ 5 \quad \text{D} \ 3 \]
1. A trucking company wants to purchase a ramp to use when loading heavy objects onto a truck. The closest that the truck can get to the loading area is 5 meters. The height from the ground to the bed of the truck is 3 meters. To the nearest meter, what should the length of the ramp be? (Lesson 1-3)

- 4 m
- 5 m
- 6 m
- 7 m

2. Which of the following is the contrapositive of the statement below? (Lesson 2-3)

If an astronaut is in orbit, then he or she is weightless.

- A If an astronaut is weightless, then he or she is in orbit.
- B If an astronaut is not in orbit, then he or she is not weightless.
- C If an astronaut is on Earth, then he or she is weightless.
- D If an astronaut is not weightless, then he or she is not in orbit.

3. Rectangle $QRST$ measures 7 centimeters long and 4 centimeters wide. Which of the following could be the dimensions of a rectangle similar to rectangle $QRST$? (Lesson 6-2)

- A 28 cm by 14 cm
- B 21 cm by 12 cm
- C 14 cm by 4 cm
- D 7 cm by 8 cm

4. A 24 foot ladder, leaning against a house, forms a 60° angle with the ground. How far up the side of the house does the ladder reach? (Lesson 7-3)

- A $12$ ft
- B $12\sqrt{2}$ ft
- C $12\sqrt{3}$ ft
- D $20$ ft

5. In rectangle $JKLM$ shown below, $\overline{JK}$ and $\overline{KM}$ are diagonals. If $JL = 2x + 5$ and $KM = 4x - 11$, what is $x$? (Lesson 8-4)

- A 10
- B 8
- C 6
- D 5

6. Joaquin bought a set of stencils for his younger sister. One of the stencils is a quadrilateral with perpendicular diagonals that bisect each other, but are not congruent. What kind of quadrilateral is this piece? (Lesson 8-5)

- A square
- B rectangle
- C rhombus
- D trapezoid

7. In the diagram below, $ABCD$ is a trapezoid with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at point $E$.

Which statement is true? (Lesson 8-6)

- A $\overline{AB}$ is parallel to $\overline{CD}$.
- B $\angle ADC$ is congruent to $\angle BCD$.
- C $\overline{CE}$ is congruent to $\overline{DE}$.
- D $\overline{AC}$ and $\overline{BD}$ bisect each other.
Chapter 8  Standardized Test Practice  459

Part 2  Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

8. At what point does the graph of $y = -4x + 5$ cross the x-axis on a coordinate plane?  (Prerequisite Skill)

9. Candace and Julio are planning to see a movie together. They decide to meet at the house that is closer to the theater. From the locations shown on the diagram, whose house is closer to the theater?  (Lesson 5-3)

10. In the diagram, $\overline{CE}$ is the mast of a sailboat with sail $\triangle ABC$.

Marcia wants to calculate the length, in feet, of the mast. Write an equation in which the geometric mean is represented by $x$.  (Lesson 7-1)

Part 3  Extended Response

Record your answers on a sheet of paper. Show your work.

12. On the tenth hole of a golf course, a sand trap is located right before the green at point $M$. Matt is standing 126 yards away from the green at point $N$. Quintashia is standing 120 yards away from the beginning of the sand trap at point $Q$.

a. Explain why $\triangle MNR$ is similar to $\triangle PQR$.  (Lesson 6-3)

b. Write and solve a proportion to find the distance across the sand trap, $a$.  (Lesson 6-3)

13. Quadrilateral $ABCD$ has vertices with coordinates: $A(0, 0)$, $B(a, 0)$, $C(a + b, c)$, and $D(b, c)$.

a. Position and label $ABCD$ on the coordinate plane. Prove that $ABCD$ is a parallelogram.  (Lesson 8-2 and 8-7)

b. If $a^2 = b^2 + c^2$, what can you determine about the slopes of the diagonals $\overline{AC}$ and $\overline{BD}$?  (Lesson 8-7)

c. What kind of parallelogram is $ABCD$?  (Lesson 8-7)